Schwarz-Christoffel Mapping

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- These mappings are defined by holomorphic functions, meaning they locally preserve angles and shape.
- Specifically, a mapping must satisfy the Cauchy-Riemann equations ($u_x = v_y$ and $u_y = -v_x$) to be complex differentiable.
- Introducing the Schwarz-Christoffel formula, a powerful tool to map the upper half-plane onto any polygon!

Conformal Mappings

Definition: Conformal Mapping

Let U and V be open subsets of $\mathbb{C}$. A function $f : U \to V$ is called conformal at a point $z_0 \in U$ if it preserves angles between directed curves through z_0, as well as their orientation. If f is a bijective holomorphic function, then its inverse is also holomorphic, and it is conformal everywhere in U.

Riemann Mapping Theorem

Theorem: Riemann Mapping Theorem

Let U be a non-empty, simply connected, open proper subset of the complex plane $\mathbb{C}$. Then there exists a bijective holomorphic mapping $f : U \to \mathbb{D}$ from U onto the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.

The Schwarz-Christoffel Formula

Derivative of the Mapping

Let P be a polygon in the complex plane with vertices $w_1, \ldots, w_n$ and corresponding interior angles $\alpha_1\pi, \ldots, \alpha_n\pi$. A conformal mapping f from the upper half-plane $\mathbb{H}$ to the interior of P has a derivative of the form:

$$f'(z) = C \prod_{k=1}^{n} (z - z_k)^{\alpha_k - 1}$$

where $z_1 < z_2 < \cdots < z_n$ are the prevertices on the real axis, and C is a complex constant.

Integrating the Map

Schwarz-Christoffel Integral

By integrating the derivative, the mapping function $f(z)$ is obtained:

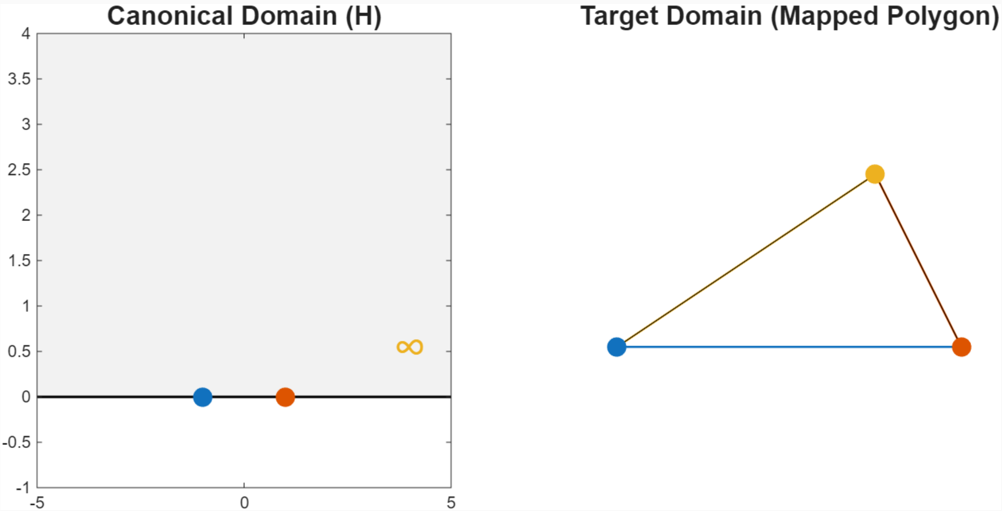
$$f(z) = C \int_{z_0}^{z} \prod_{k=1}^{n} (\zeta - z_k)^{\alpha_k - 1} d\zeta + c_1$$

where c_1 is a constant of integration representing a translation in the complex plane, and C represents a combined rotation and scaling factor. The real constants z_k (the prevertices) must be determined to properly match the target polygon.

Schwarz-Christoffel

- SC maps essentially unfold the real axis onto the boundary of the target polygon.
- The prevertices z_k act as the hinges where the boundary abruptly bends to form the polygon's vertices w_k.
- The exponent $(\alpha_k - 1)$ determines the turning angle at each vertex.
- As long as the interior angles $\alpha_k \pi$ sum to $(n-2)\pi$, the polygon will close correctly.
- Toy example: Mapping the upper half-plane to a rectangle requires 4 prevertices. The integral becomes an elliptic integral of the first kind, which usually requires numerical evaluation to construct the domain.

Schwarz-Christoffel Mapping



Review of the Upper Half-Plane

- **Intuition.** The upper half-plane $\mathbb{H} = \{z \in \mathbb{C} : \operatorname{Im}(z) > 0\}$ is a standard canonical domain in complex analysis.

Properties of $\mathbb{H}$ and the Real Line:

1. The boundary of $\mathbb{H}$ is the real axis $\mathbb{R}$ combined with the point at infinity.
2. A continuous walk along $\mathbb{R}$ corresponds to a perimeter walk along the target polygon.
3. Passing a prevertex z_k causes the direction of the boundary curve in the target domain to change by a specific angle.

- The mapping f is **conformal** everywhere in the interior of $\mathbb{H}$, but the derivative $f'(z)$ becomes either zero or infinite precisely at the prevertices z_k.

Degrees of Freedom in SC Maps

The Parameter Problem. A conformal mapping from $\mathbb{H}$ to a polygon is not uniquely defined without fixing some parameters:

- By the Riemann Mapping Theorem, we have 3 real degrees of freedom.
- This allows us to arbitrarily choose the positions of three prevertices on the real line (e.g., $z_1 = -1$, $z_2 = 0$, $z_3 = 1$, or placing one at ∞).
- For polygons with $n > 3$ vertices, the remaining $n - 3$ prevertices must be determined iteratively.

This non-linear root-finding problem is historically known as the **Schwarz-Christoffel parameter problem**.

Toy Construction: Mapping to a Triangle

Step 1: Define the Polygon

Let P be a triangle with vertices w_1, w_2, w_3. By definition, $n = 3$.

- Interior angles: $\alpha_1\pi, \alpha_2\pi, \alpha_3\pi$ where $\alpha_1 + \alpha_2 + \alpha_3 = 1$.
- Because $n = 3$, we can freely choose all three prevertices without any iterative solving.

We typically map one prevertex to infinity to simplify the resulting integral.

Step 2: Choose Prevertices

Let $z_1 = -1$, $z_2 = 1$, and $z_3 = \infty$. Because z_3 is at infinity, its term drops out of the differential equation entirely. The SC derivative becomes:

$$f'(z) = C(z+1)^{\alpha_1 - 1}(z-1)^{\alpha_2 - 1}$$

We can notice how the complexity collapses completely when $n \leq 3$.

Toy Construction: Mapping to a Triangle (Cont.)

Step 3: Integration

To find the exact mapping to the triangle, we integrate the derivative:

- $f(z) = C \int_0^z (\zeta + 1)^{\alpha_1 - 1} (\zeta - 1)^{\alpha_2 - 1} d\zeta + c_1$

From integrations along the segments to $z_1 = -1$ and $z_2 = 1$, we can map the real axis segments:

$$w_1 = f(-1), \quad w_2 = f(1)$$

The complex constants C and c_1 are determined by solving a simple linear system:

$$w_1 = CI_1 + c_1 \quad \text{and} \quad w_2 = CI_2 + c_1$$

where I_1 and I_2 are the evaluated definite integrals.

Conclusion: For a triangle, no iterative parameter guessing is required. The map is determined analytically (often yielding standard Beta functions or hypergeometric functions).

What about $n > 3$?

The Challenge with Rectangles ($n = 4$)

- We can fix 3 prevertices (e.g., $-1, 1, \infty$), but the 4th prevertex z_4 is unknown.
- The integral becomes an *elliptic integral*, which links to profound topics in algebraic geometry.

The Crowding Phenomenon

- When mapping elongated polygons, prevertices on the real axis crowd exponentially close together.
- A microscopic change in a prevertex position causes a massive distortion in the mapped polygon.
- Precision loss can be catastrophic, requiring robust numerical techniques.

Conclusion.

- While purely analytical for triangles, most practical SC maps for engineering require numerical solvers to manage crowding and the parameter problem.

Main Idea of Modern Implementations

So, how do we efficiently compute SC mappings for complex polygons without precision loss?

Key idea.

- Modify the canonical domain! Instead of the upper half-plane or unit disk, map from an elongated domain like an infinite strip or a rectangle.
- If the canonical domain naturally resembles the target polygon, the crowding phenomenon is severely mitigated.
- Modern software packages (like the SC Toolbox in MATLAB) implement robust algorithms (e.g., using Cross Ratios of the Delaunay Triangulation) to solve the parameter problem efficiently.

Laplace's Equation and SC Maps

Let $\phi(x, y)$ be a harmonic function representing electrostatic potential or steady-state heat distribution in a complex polygonal domain P. Because conformal maps preserve Laplace's equation ($\nabla^2\phi = 0$), we can map P to the upper half-plane $\mathbb{H}$ using the inverse SC map, solve the simpler boundary value problem in $\mathbb{H}$, and map the solution back to P.

Generic Transform: Domain Mapping

Idea of the transform:

- Keep the original physical boundary conditions (e.g., Dirichlet or Neumann).
- Map the complex polygonal geometry to a canonical domain (like $\mathbb{D}$ or $\mathbb{H}$).
- **Solve the PDE on the simple domain**:

$$\nabla^2 \Phi(u, v) = 0 \text{ in } \mathbb{H}$$

Intuition.

- The geometry of the boundary is mathematically absorbed into the mapping function $f(z)$.
- The SC formula effectively flattens physical corners into straight lines, allowing standard half-plane Poisson kernels to be applied.
- Complicated physical flow or heat lines in the polygon can be perfectly visualized by mapping straight grid lines from $\mathbb{H}$.

Example: Dirichlet Problem in a Strip

1. The Physical Problem

- Let our target domain P be a semi-infinite strip in the w-plane ($w = u + iv$): $-\pi/2 \leq u \leq \pi/2$ and $v \geq 0$.
- We want to find the steady-state temperature $\phi(u, v)$ satisfying Laplace's equation, $\nabla^2\phi = 0$.
- **Dirichlet BCs:** The base segment ($v = 0$) is heated to a constant V_0. The vertical walls ($u = \pm\pi/2$) are kept cold at 0.

2. The Conformal Mapping

- The sine function, $z = \sin(w)$, maps the strip P to our canonical upper half-plane $\mathbb{H}$.
- The heated base of the strip maps strictly to the interval $[-1, 1]$ on the real axis in $\mathbb{H}$.
- The cold walls map to the rest of the real axis: $(-\infty, -1]$ and $[1, \infty)$.
- Our boundary conditions in $\mathbb{H}$ become simple: $\Phi(x, 0) = V_0$ for $x \in [-1, 1]$, and 0 elsewhere.

Example: Solving and Mapping Back

3. Solving in the Canonical Domain

- In $\mathbb{H}$, the solution for a piecewise constant boundary condition is found simply by measuring angles.
- The potential at any point z is proportional to the "viewing angle" of the heated segment $[-1, 1]$ from that point:

$$\Phi(x, y) = \frac{V_0}{\pi}\big(\arg(z + 1) - \arg(z - 1)\big)$$

4. The Inverse Transform

- To find the solution in our original physical strip P, we compose our canonical solution with the mapping $z = \sin(w)$:

$$\phi(u, v) = \frac{V_0}{\pi}\big(\arg(\sin(w) + 1) - \arg(\sin(w) - 1)\big)$$

- **Takeaway:** By leveraging conformal mappings, a complicated PDE boundary value problem was reduced to elementary geometry in the upper half-plane!

Conclusion

- Conformal mappings provide a bridge between complicated polygonal geometries and simple canonical domains.
- The Schwarz-Christoffel formula dictates exactly how the boundary must bend to form the desired vertices.
- The main computational bottleneck is the parameter problem, which is now largely solved by modern numerical toolboxes.
- Once mapped, complex physical phenomena (electrostatics, fluid dynamics) become tractable.

Essentially, the Schwarz-Christoffel mapping is a powerful tool to hide geometric complexity within a complex integral.