Vietoris–Rips Complexes of Torus Grids

by

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Abstract

This thesis serves as an exposition of the results presented in the paper [3], which I coauthored with Henry Adams, Adenike Yeside Adetowubo, Hector Barriga-Acosta, and Ziqin Feng. In this work, we study the homotopy types and homology of Vietoris–Rips complexes of torus grid graphs. Let $T_{n,n}$ be a torus grid graph consisting of $n \times n$ points on $\mathbb{T}^2$ equipped with the l^1 metric. We first compute the homology of $\mathrm{VR}(T_{n,n}; k)$ for $1 \leq n \leq 28$ and $1 \leq k \leq 13$. Subsequently, we provide a complete classification of the maximal simplices in $\mathrm{VR}(T_{3k,3k}; k)$ for $k \geq 2$, $\mathrm{VR}(T_{3k-1,3k-1}; k)$ for $k \geq 3$, and $\mathrm{VR}(T_{n,n}; k)$ when $k \geq 2$ and $n \geq 3k$. Using these classifications, we establish the homotopy equivalences: $\mathrm{VR}(T_{3k,3k}; k) \simeq \bigvee_{6k^2-1} S^2$ for $k \geq 2$, $\mathrm{VR}(T_{3k-1,3k-1}; k) \simeq \bigvee_{6k-3} S^2 \vee \bigvee_{6k-2} S^3$ for $k \geq 3$, and $\mathrm{VR}(T_{n,n}; k) \simeq \mathbb{T}^2$ when $k \geq 2$ and $n \geq 3k$.

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Chapter 1

Introduction

Given any metric space X and scale $\epsilon \geq 0$, we define the Vietoris–Rips complex on X at scale ϵ, denoted $\mathrm{VR}(X;\epsilon)$, to be the simplicial complex whose vertex set is X and simplices are finite subsets $\sigma \subseteq X$ with diameter not exceeding ϵ.

Vietoris–Rips complexes were created by Leopold Vietoris [19] in an attempt to extend early homology and cohomology theories to general metric spaces. In the late 20th century, Mikhail Gromov [13] and Elihayu Rips found applications for Vietoris–Rips complexes in geometric group theory, particularly in the study of hyperbolic groups. More recently, following algorithmic improvements in the computation of persistent homology [4], Vietoris–Rips complexes have been used at small scales to study point cloud data in applied algebraic topology and topological data analysis, see for instance [7, 10]. Vietoris-Rips complexes have also found applications in a variety of domains, such as medical imaging, computer vision, and chemical physics, see for example [5, 8, 17].

Motivation for the study of Vietoris-Rips complexes is driven largely by foundational results of Hausmann and Latschev, which establish when $\mathrm{VR}(M;\epsilon)$ recovers the homotopy type of a Riemannian manifold M. Specifically, Hausmann showed that $\mathrm{VR}(M;\epsilon)$ is homotopy equivalent to M for ϵ sufficiently small [15, Theorem 3.5]. Later, Latschev extended this by proving that, for a metric space Y, $\mathrm{VR}(Y;\epsilon)$ is homotopy equivalent to M when Y is Gromov-Hausdorff close to M, for M a closed Riemannian manifold [16, Theorem 1.1].

As such, in recent years, much work has gone towards understanding the topology of Vietoris-Rips complexes of a variety of spaces. In particular, the homotopy types and homology groups of Vietoris-Rips complexes have been either entirely or partially classified for the circle [2], integer lattices [20], hypercube graphs [11], planar point sets [9], geodesic spaces [21], and Platonic solids [18].

	$k=1$	2	3	4	5	6	7	8	9
$n=3$	$\bigvee^{4} S^1$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$
4	$\bigvee^{17} S^1$	$\bigvee^{9} S^3$	S^7	$*$	$*$	$*$	$*$	$*$	$*$
5	$\bigvee^{26} S^1$	$\bigvee^{9} S^2$	$\bigvee^{9} S^4$	$*$	$*$	$*$	$*$	$*$	$*$
6	$\bigvee^{37} S^1$	$\bigvee^{23} S^2$	3:1, 5:12	5:23, 8:2	S^{17}	$*$	$*$	$*$	$*$
7	$\bigvee^{50} S^1$	$\mathbb{T}^2$	3:1, 4:14	S^3	11:1	$*$	$*$	$*$	$*$
8	$\bigvee^{65} S^1$	$\mathbb{T}^2$	$\bigvee^{15} S^2 \vee \bigvee^{16} S^3$	3:1, 7:16 (12:0)	3:1 (12:0)	12:259	S^{31}	$*$	$*$
9	$\bigvee^{82} S^1$	$\mathbb{T}^2$	$\bigvee^{53} S^2$	3:1, 5:36 (9:0)	3:1 (9:0)	7:31, 8:183 (9:0)	(6:0)	$*$	$*$
10	$\bigvee^{101} S^1$	$\mathbb{T}^2$	$\mathbb{T}^2$	3:21, 4:60	3:1, 9:20	4:39 (10:0)	(6:0)	(6:0)	S^{49}
11	$\bigvee^{122} S^1$	$\mathbb{T}^2$	$\mathbb{T}^2$	$\bigvee^{21} S^2 \vee \bigvee^{22} S^3$	3:1, 6:22	3:1 (10:0)	4:1, 5:4	(5:0)	(5:0)
12	$\bigvee^{145} S^1$	$\mathbb{T}^2$	$\mathbb{T}^2$	$\bigvee^{95} S^2$	3:1, 4:24, 5:120	3:1 (9:0)	3:1 (6:0)	(4:0)	(4:0)
13	$\bigvee^{170} S^1$	$\mathbb{T}^2$	$\mathbb{T}^2$	$\mathbb{T}^2$	3:27, 4:26	3:1, 8:26	3:1 (6:0)	3:1 (4:0)	(4:0)
14	$\bigvee^{197} S^1$	$\mathbb{T}^2$	$\mathbb{T}^2$	$\mathbb{T}^2$	$\bigvee^{27} S^2 \vee \bigvee^{28} S^3$	3:1, 5:28, 6:84	3:1 (6:0)	3:1 (6:0)	4:1
15	$\bigvee^{226} S^1$	$\mathbb{T}^2$	$\mathbb{T}^2$	$\mathbb{T}^2$	$\bigvee^{149} S^2$	3:1, 4:90	3:1 (4:0)	3:1 (4:0)	4:89
16	$\bigvee^{257} S^1$	$\mathbb{T}^2$	$\mathbb{T}^2$	$\mathbb{T}^2$	$\mathbb{T}^2$	3:35, 4:34 (6:0)	3:1, 6:64 (7:0)	3:1 (6:0)	3:1 (4:0)
17	$\bigvee^{290} S^1$	$\mathbb{T}^2$	$\mathbb{T}^2$	$\mathbb{T}^2$	$\mathbb{T}^2$	$\bigvee^{33} S^2 \vee \bigvee^{34} S^3$	3:1, 4:34, 5:34	3:1 (6:0)	3:1 (4:0)
18	$\bigvee^{325} S^1$	$\mathbb{T}^2$	$\mathbb{T}^2$	$\mathbb{T}^2$	$\mathbb{T}^2$	$\bigvee^{215} S^2$	3:3, 4:2, 5:216	3:1 (6:0)	3:1 (4:0)
19	$\bigvee^{362} S^1$	$\mathbb{T}^2$	$\mathbb{T}^2$	$\mathbb{T}^2$	$\mathbb{T}^2$	$\mathbb{T}^2$	3:39, 4:38 (5:0)	3:1 (4:0)	3:1 (4:0)
20	$\bigvee^{401} S^1$	$\mathbb{T}^2$	$\mathbb{T}^2$	$\mathbb{T}^2$	$\mathbb{T}^2$	$\mathbb{T}^2$	$\bigvee^{39} S^2 \vee \bigvee^{40} S^3$	3:1, 4:160 (5:0)	3:1 (5:0)
21	$\bigvee^{442} S^1$	$\mathbb{T}^2$	$\mathbb{T}^2$	$\mathbb{T}^2$	$\mathbb{T}^2$	$\mathbb{T}^2$	$\bigvee^{293} S^2$	3:3, 4:2, 5:294	3:1 (5:0)
22	$\bigvee^{485} S^1$	$\mathbb{T}^2$	$\mathbb{T}^2$	$\mathbb{T}^2$	$\mathbb{T}^2$	$\mathbb{T}^2$	$\mathbb{T}^2$	3:47, 4:46 (5:0)	3:1, 4:44, 5:44
23	$\bigvee^{530} S^1$	$\mathbb{T}^2$	$\mathbb{T}^2$	$\mathbb{T}^2$	$\mathbb{T}^2$	$\mathbb{T}^2$	$\mathbb{T}^2$	$\bigvee^{45} S^2 \vee \bigvee^{46} S^3$	3:3, 4:2 (5:0)
24	$\bigvee^{577} S^1$	$\mathbb{T}^2$	$\mathbb{T}^2$	$\mathbb{T}^2$	$\mathbb{T}^2$	$\mathbb{T}^2$	$\mathbb{T}^2$	$\bigvee^{383} S^2$	3:3, 4:2, 5:384
25	$\bigvee^{626} S^1$	$\mathbb{T}^2$	$\mathbb{T}^2$	$\mathbb{T}^2$	$\mathbb{T}^2$	$\mathbb{T}^2$	$\mathbb{T}^2$	$\mathbb{T}^2$	3:51, 4:50 (5:0)
26	$\bigvee^{677} S^1$	$\mathbb{T}^2$	$\mathbb{T}^2$	$\mathbb{T}^2$	$\mathbb{T}^2$	$\mathbb{T}^2$	$\mathbb{T}^2$	$\mathbb{T}^2$	$\bigvee^{51} S^2 \vee \bigvee^{52} S^3$
27	$\bigvee^{730} S^1$	$\mathbb{T}^2$	$\mathbb{T}^2$	$\mathbb{T}^2$	$\mathbb{T}^2$	$\mathbb{T}^2$	$\mathbb{T}^2$	$\mathbb{T}^2$	$\bigvee^{485} S^2$
28	$\bigvee^{785} S^1$	$\mathbb{T}^2$	$\mathbb{T}^2$	$\mathbb{T}^2$	$\mathbb{T}^2$	$\mathbb{T}^2$	$\mathbb{T}^2$	$\mathbb{T}^2$	$\mathbb{T}^2$

Table 1.1: Homotopy types of $\mathrm{VR}(T_{n,n}; k)$. We write $i{:}d$ as shorthand for $H_i(\mathrm{VR}(T_{n,n}; k); \mathbb{Z}/2) \cong (\mathbb{Z}/2)^d$. Cells that contain an entry of the form (i:0) have been computed up to the i-th homology group without yielding additional features. Cells not containing (i:0) have had the homology computed up to the greatest i within the cell. All computations were done on Auburn University's Easley cluster using $\mathbb{Z}/2$ coefficients in Ripser [4]. Each distinct colored diagonal corresponds to a major result discussed in [3]: the blue-highlighted diagonal is explained in [3, Section 6], while the red and teal diagonals are explained in Section 4.

This thesis aims to clarify a recent continuation of this tradition in [3] that partially classifies the homotopy types of $\mathrm{VR}(T_{n,n}; \epsilon)$ where $T_{n,n}$ is an $n \times n$ torus grid graph equipped with the l^1 metric. Part of the motivation for studying $T_{n,n}$ with the l^1 metric is a number of conjectures arising from homology computations of $\mathrm{VR}(T_{n,n};\epsilon)$, see Table 1.1 for known homotopy types and computed homology groups. For example, the conjecture leading to the result proving $\mathrm{VR}(T_{3k,3k};k) \simeq \vee^{6k^2-1} S^2$ in Chapter 4 was born out of homology computations asserting $H_2(\mathrm{VR}(T_{3k,3k};k);\mathbb{Z}/2) \cong (\mathbb{Z}/2)^{6k^2-1}$. For an explanation of column 1, the blue-highlighted diagonal, and the contractibility results, we direct the reader to [3].

As the main results of this thesis, we prove that:

1. $\mathrm{VR}(T_{n,n};k) \simeq \mathbb{T}^2$ for any $k \geq 3$ and $n > 3k$,
2. $\mathrm{VR}(T_{3k,3k};k) \simeq \bigvee_{6k^2-1} S^2$ for any $k \geq 2$, and
3. $\mathrm{VR}(T_{3k-1,3k-1};k) \simeq \bigvee_{6k-3} S^2 \vee \bigvee_{6k-2} S^3$ for any $k \geq 3$.

The first result is obtained by first classifying the maximal simplices of $\mathrm{VR}(T_{n,n};k)$ under the given conditions. We then construct an open cover of $\mathbb{T}^2$ whose nerve complex is homotopy equivalent to $\mathbb{T}^2$. Lastly, we apply a result from Adams et al. [3] to show that this nerve is homotopy equivalent to the complex with the same maximal simplices as $\mathrm{VR}(T_{n,n};k)$, yielding the result.

The second result is obtained by decomposing $\mathrm{VR}(T_{3k,3k};k)$ into two subcomplexes. From the first result, it will be immediate that one of these subcomplexes is homotopy equivalent to $\mathbb{T}^2$. Following this, we note that the other subcomplex consists of $6k^2$ 2-simplices such that the intersection of any two such simplices is at most a vertex. Using the homotopy equivalence of 2-simplices and discs, we attach $3k^2$ discs to $\mathbb{T}^2$ with their boundary circles attached longitudinally and the other $3k^2$ discs are attached along their boundaries horizontally such that they cover the central hole of $\mathbb{T}^2$. By induction, this yields that $\mathrm{VR}(T_{3k,3k};k) \simeq \bigvee_{6k^2-1} S^2$.

Finally, the third result is obtained in a manner analogous to the second. We begin by decomposing $\mathrm{VR}(T_{3k-1,3k-1};k)$ into two subcomplexes. As in the second result, one of these is homotopy equivalent to $\mathbb{T}^2$. The other consists of $2(3k-1)^2$ tetrahedra, which we further divide into $6k-2$ subcomplexes of $\mathrm{VR}(T_{3k-1,3k-1};k)$. By a result of Adamaszek from [1, Corollary 6.7], each of these $6k-2$ subcomplexes is homotopy equivalent to S^3. We then show that attaching these subcomplexes sequentially to $\mathbb{T}^2$ contributes a wedge summand of $S^2 \vee S^3$ at each step. Inductively, this yields the final homotopy equivalence: $\mathrm{VR}(T_{3k-1,3k-1};k) \simeq \bigvee_{6k-3} S^2 \vee \bigvee_{6k-2} S^3$.

Additionally, based on the above homology computations, we conjecture that for $n \geq 3$ the reduced homology of $\mathrm{VR}(T_{3n-2,3n-2};n)$ is nontrivial only in dimensions 3 and 4.

We start with preliminaries in Chapter 2. Following this, in Chapter 3, we classify the maximal simplices for $\mathrm{VR}(T_{n,n};k)$ when $k \geq 3$ and $n > 3k$, $\mathrm{VR}(T_{3k,3k};k)$ when $k \geq 2$, and $\mathrm{VR}(T_{3k-1,3k-1};k)$ when $k \geq 3$. We conclude in Chapter 4 by using our maximal simplices classifications to show that $\mathrm{VR}(T_{n,n};k) \simeq \mathbb{T}^2$ when $k \geq 3$ and $n > 3k$, $\mathrm{VR}(T_{3k,3k};k) \simeq \bigvee_{6k^2-1} S^2$ for $k \geq 2$, and $\mathrm{VR}(T_{3k-1,3k-1};k) \simeq \bigvee_{6k-3} S^2 \vee \bigvee_{6k-2} S^3$ for $k \geq 3$.

Chapter 2

Preliminaries

In this chapter, we present definitions, theorems, and notation related to metric spaces, graphs, simplicial complexes, and general topology that will be used throughout the remainder of this work.

2.1 Metric Spaces

The primary metric space we are concerned with in this paper is the $\mathbf{l^1}$ **metric**, also called the **Manhattan metric**, on $\mathbb{Z}^2$ or $T_{n,n}$ where

$$d((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| + |y_1 - y_2|.$$

For convenience, we let d represent the l^1 metric on both $\mathbb{Z}^2$ and $T_{n,n}$.

We denote **open balls** and **closed balls** in the metric space X, centered at $x \in X$ with radius $r \geq 0$, by

$$B_X(x, r) = \{y \in X : d(x, y) < r\} \qquad \text{and} \qquad B_X[x, r] = \{y \in X : d(x, y) \leq r\},$$

respectively. When there is no ambiguity in the metric space X, we omit the subscript and write $B(x, r)$ for open balls and $B[x, r]$ for closed balls.

Finally, the **diameter** of a subset $X \subseteq Y$ is defined as $\text{diam}(X) = \sup\{d(a, b) : a, b \in X\}$. If $\text{diam}(X)$ is not finite, then we set $\text{diam}(X) = \infty$.

2.2 Graphs

A **graph** is a pair $G = (V(G), E(G))$, where $V(G)$ is the *vertex set*, consisting of elements called *vertices*, and $E(G)$ is the *edge set*, consisting of unordered pairs $\{v, w\}$ with $v, w \in V(G)$. A **clique** in G is a subset $C \subseteq V(G)$ such that $\{v, w\} \in E(G)$ for all distinct $v, w \in C$. A graph G is said to be **connected** if for every pair of vertices $v, w \in V(G)$, there exists a path (a sequence of edges) connecting v and w.

For a connected graph G, the **shortest path metric** is the function $d_G \colon V(G) \times V(G) \to \mathbb{N}$ defined by letting $d_G(v, w)$ be the length of the shortest path between v and w in G. This definition extends to arbitrary graphs by setting $d_G(v, w) = \infty$ if there is no path from v to w.

Given a graph G and a positive integer k, the **k-th power** of G, denoted G^k, is the graph with vertex set $V(G^k) = V(G)$ and edge set $E(G^k) = \big\{\{v, w\} : d_G(v, w) \le k\big\}$.

For an integer $n \ge 3$, the **cycle graph** C_n is the graph with vertex and edge set

$$V(C_n) = \{0, 1, \dots, n-1\}$$
$$E(C_n) = \big\{\{i, i+1\} : 0 \le i \le n-2\big\} \cup \big\{\{n-1, 0\}\big\}.$$

Given graphs $G = (V(G), E(G))$ and $H = (V(H), E(H))$, the **Cartesian product** $G \square H$ is the graph defined as follows:

1. $V(G \square H) = V(G) \times V(H)$,
2. Two vertices $(v, v'), (u, u') \in V(G \square H)$ are adjacent if and only if either
   - $v = u$ and $\{v', u'\} \in E(H)$, or
   - $v' = u'$ and $\{v, u\} \in E(G)$.

2.3 Torus Grid Graphs

Let $T_{n,n}$ be an $n \times n$ graph on a torus, that is $T_{n,n} \subseteq S^1 \times S^1$. While this definition is entirely sufficient for intuition, we work with the following construction.

Definition 2.1. Let $G_n = n\mathbb{Z} \times n\mathbb{Z}$ where $n\mathbb{Z} = \{nm : m \in \mathbb{Z}\}$. The group G_n acts on $\mathbb{Z}^2$ by $g \cdot (x, y) = (x + i, y + j)$ where $g = (i, j) \in n\mathbb{Z} \times n\mathbb{Z}$. By construction, it is clear that $T_{n,n}$ is isometric to $\mathbb{Z}^2/G_n$ equipped with the quotient metric. Thus, we define a **torus grid graph** as the quotient $\mathbb{Z}^2/G_n$, where $G_n = n\mathbb{Z} \times n\mathbb{Z}$.

As noted above, we use d to represent the l^1 metric on both $\mathbb{Z}^2$ and the quotient $\mathbb{Z}^2/G_n$. Denote $\pi_n(x, y) = ([x], [y])$ where π_n is the quotient map $\mathbb{Z}^2 \to \mathbb{Z}^2/G_n$. The metric on $\mathbb{Z}^2/G_n$ satisfies $d(([x], [y]), ([w], [z])) = \min\{d((x', y'), (w', z')) : (x', y') \in ([x], [y])$ and $(w', z') \in ([w], [z])\}$. Moreover, for any two points $(x', y'), (x'', y'') \in ([x], [y])$, we have $d((x', y'), (x'', y'')) \geq n$.

More generally, the torus grid graph $T_{n,m}$ can be defined as the Cartesian product $C_n \square C_m$.

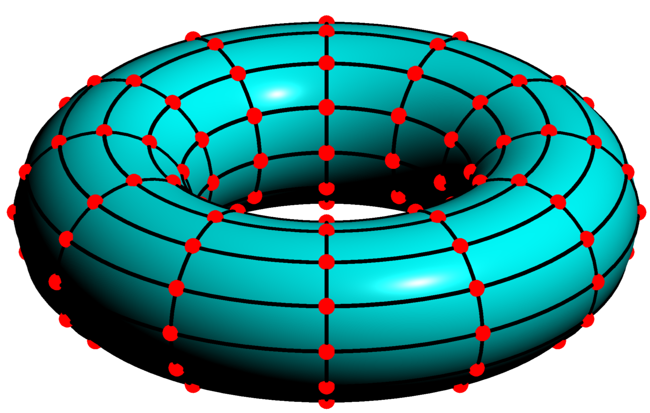


Figure 2.1: 12×12 torus grid highlighted by the red vertices and black edges.

2.4 Simplicial complexes

Given a set $V = \{v_0, v_1, \ldots, v_k\}$ in $\mathbb{R}^{k+1}$, an **affine combination** is a linear combination

$$\sum_{i=0}^{k} a_i v_i \quad \text{such that} \quad \sum_{i=0}^{k} a_i = 1.$$

The **convex hull** of V is the set of all affine combinations with $a_i \geq 0$ for all i. The set V is **affinely independent** if no point in V is an affine combination of points in V. Equivalently, V is affinely independent if whenever $\sum_{i=0}^{k} a_i = 0$ and $\sum_{i=0}^{k} a_i v_i = 0$, it follows that $a_i = 0$ for all i.

A **k-simplex** σ is the convex hull of $k+1$ affinely independent points in $\mathbb{R}^{k+1}$. If V is affinely independent, then the k-simplex τ determined by V is the convex hull of its points, and each element of V is a vertex of τ. A **face** of σ is a simplex τ satisfying $\tau \subseteq \sigma$.

The standard k-simplex in $\mathbb{R}^{k+1}$ is the convex hull of the points $(a_i)_{i=0}^{k}$ where $a_n = 1$ for exactly one $n \in [k]$ and is zero elsewhere. These points are the standard basis vectors of $\mathbb{R}^{k+1}$, and they form the vertices of the standard k-simplex. Geometrically, a 0-simplex is a point, a 1-simplex is a line segment, a 2-simplex is a triangle, and a 3-simplex is a tetrahedron.

A **simplicial complex** K in $\mathbb{R}^{k+1}$ is a finite set of simplices satisfying:

1. Every face of a simplex σ in K is in K, and
2. Given two simplices $\sigma, \tau \in K$, either $\sigma \cap \tau = \emptyset$ or $\sigma \cap \tau \subseteq \sigma$ and $\sigma \cap \tau \subseteq \tau$.

An **abstract simplicial complex** K consists of a set of vertices, say V_K, and a set S_K of nonempty subsets of V_K called simplices satisfying the following:

1. For any $\sigma \subseteq V_K$, if $\tau \subseteq \sigma$, then $\tau \in S_K$, and
2. For each $v \in V_K$, $\{v\} \in S_K$.

The simplicial complex K_1 is a **subcomplex** of K, denoted $K_1 \subseteq K$, if $V_{K_1} \subseteq V_K$ and $S_{K_1} \subseteq S_K$. A subcomplex K_1 is **full** in K if for every $\sigma \in S_K$ such that $\sigma \subseteq V_{K_1}$, we have $\sigma \in S_{K_1}$.

Given a graph $G = (V(G), E(G))$, the **clique complex** of the graph G, denoted $\mathrm{Cl}(G)$, is the simplicial complex with vertex set $V(G)$, and $\sigma \subset V(G)$ a simplex if and only if σ is a clique.

2.5 CW complexes

A **standard unit ball** in $\mathbb{R}^d$ is defined to be $B^d = \{x \in \mathbb{R}^d : \|x\| \leq 1\}$ where $\|\cdot\|$ denotes the standard Euclidean norm. The **standard unit sphere** is the boundary of B^d, given by $S^{d-1} = \{x \in \mathbb{R}^d : \|x\| = 1\}$.

A finite-dimensional **CW complex** X is a topological space constructed by successively attaching cells of increasing dimension. Formally, X is the result of a sequence of topological spaces

$$X_0 \subseteq X_1 \subseteq X_2 \subseteq \cdots \subseteq X_n = X,$$

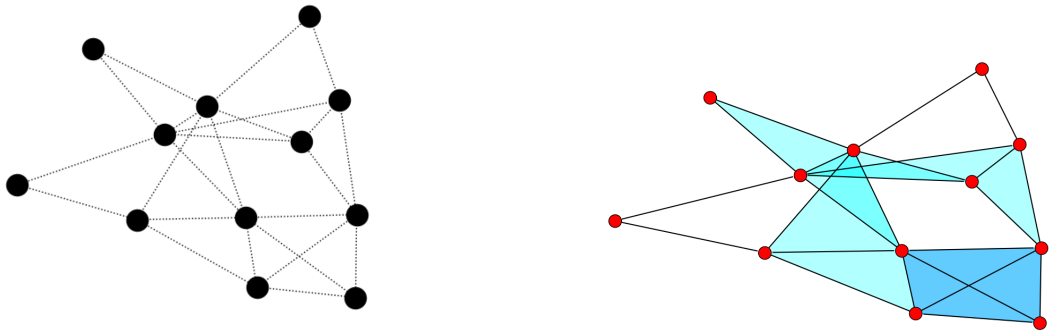
where each X_i is called the i-skeleton of X. The construction of X begins with X_0, a discrete set of points. For each $i \geq 1$, the i -skeleton X_i is obtained from X_{i-1} by attaching i-dimensional balls B^i to X_{i-1} via gluing maps $\gamma : S^{i-1} \to X_{i-1}$. The interiors of the attached balls form the i-cells of X.

2.6 Vietoris–Rips complexes

Definition 2.2. Let X be a metric space with distance function d. The **Vietoris–Rips complex** with parameter $\epsilon \geq 0$, denoted $\mathrm{VR}((X, d); \epsilon)$ or simply $\mathrm{VR}(X; \epsilon)$, is the simplicial complex defined as follows:

1. X is the vertex set of $\mathrm{VR}((X, d); \epsilon)$, and

2. A finite subset $\sigma \subseteq X$ is the vertex set of a simplex if $\mathrm{diam}(\sigma) \leq \epsilon$.



(a) Underlying graph G **(b)** $\mathrm{VR}((G, d_G); 1)$

Figure 2.2: Vietoris-Rips complex of a graph G equipped with the shortest path metric. The red vertices are 0-simplices, the black edges are 1-simplices, the light blue triangles are 2-simplices, and the dark blue tetrahedra are 3-simplices.

Given a graph $G = (V(G), E(G))$ equipped with the shortest path metric d_G, we observe that the clique complex of G^k, denoted $\mathrm{Cl}(G^k)$, is equivalent to the Vietoris-Rips complex $\mathrm{VR}((V(G), d_G); k)$.

2.7 Nerve complexes

Let X be a topological space. Given a finite collection of sets $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ with each $U_\alpha \subseteq X$, the **nerve complex** of $\mathcal{U}$ is the simplicial complex $N(\mathcal{U})$ whose vertex set is the index set A, and where a subset $\{\alpha_0, \alpha_1, ..., \alpha_k\} \subseteq A$ spans a k-simplex in $N(\mathcal{U})$ if and only if $U_{\alpha_0} \cap U_{\alpha_1} \cap ... \cap U_{\alpha_k} \neq \emptyset$.

Theorem 2.3 (Nerve Theorem)**.** *Let $\mathcal{U}$ be a collection of open sets in a metric space X with $\cup\mathcal{U} = X$. If every nonempty intersection $\bigcap_{i=0}^{n} U_{\alpha_i}$ is contractible, then the nerve simplicial complex $N(\mathcal{U})$ is homotopy equivalent to X.*

This version of the nerve theorem follows from [14, Corollary 4G.3] since every metric space is paracompact.

2.8 Homology

Given a simplicial complex K, we define $\Delta_n(K)$ to be the free abelian group consisting of finite integral combinations of the oriented n-simplices in K. The **boundary homomorphism** $\partial_n : \Delta_n(K) \to \Delta_{n-1}(K)$ is a function that maps each oriented n-simplex to a linear combination of its $(n-1)$-dimensional faces, each with an appropriate sign to respect orientation, such that $\partial_{n-1} \circ \partial_n = 0$. The collection of all $\Delta_i(K)$ connected by boundary homomorphisms, ∂_i, form a **chain complex**

$$\cdots \xrightarrow{\partial_{n+2}} \Delta_{n+1}(K) \xrightarrow{\partial_{n+1}} \Delta_n(K) \xrightarrow{\partial_n} \Delta_{n-1}(K) \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} \Delta_1(K) \xrightarrow{\partial_1} \Delta_0(K) \to 0$$

where $\partial_i \circ \partial_{i+1} = 0$ for each i.

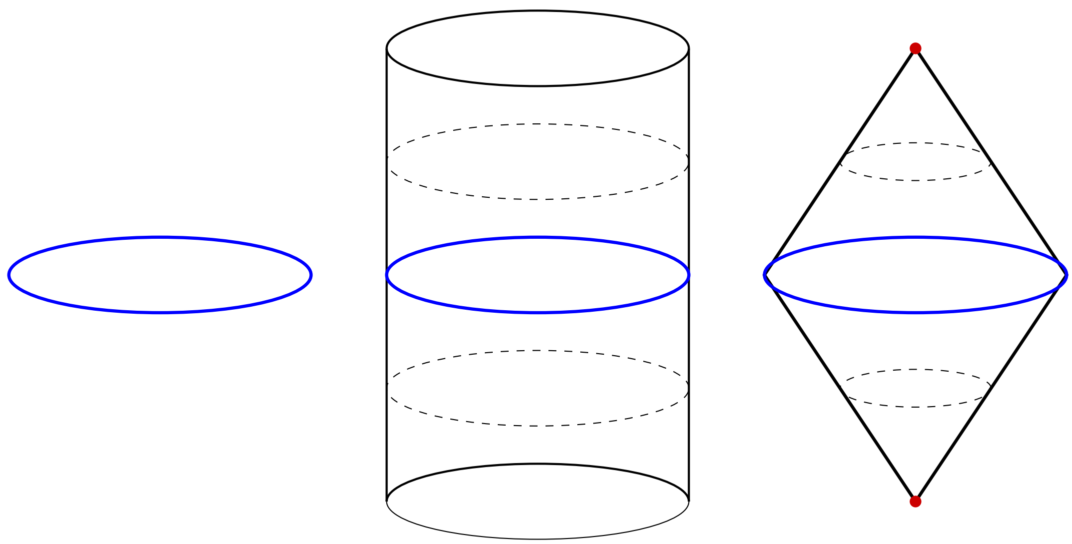
The ***i*-th homology group** (with $\mathbb{Z}$ coefficients), $H_i(K)$, is defined to be the quotient group $\mathrm{Ker}\partial_i/\mathrm{Im}\partial_{i+1}$. Moreover, the ***i*-th Betti number** $\beta_i(K)$ is the rank of $H_i(K)$, i.e. $\beta_i(K) = \mathrm{rank}(H_i(K))$.

2.9 Homotopy types

Given two continuous maps $f, f' : X \to Y$, we say that f is **homotopic** to f', written $f \simeq f'$, if there exists a continuous map $F : X \times [0,1] \to Y$ such that $F(x,0) = f(x)$ and $F(x,1) = f'(x)$. The map F is called a **homotopy** between f and f'. If f' is a constant map and $f \simeq f'$, then f is said to be **nullhomotopic**.

Given two topological spaces X and Y, we say X is **homotopy equivalent** to Y, denoted $X \simeq Y$, if there exist continuous maps $f : X \to Y$ and $f' : Y \to X$ such that $f \circ f' \simeq 1_Y$ and $f' \circ f \simeq 1_X$. Intuitively, two spaces are homotopy equivalent if each space can be continuously deformed into the other. A topological space X is called **contractible** if it is homotopy equivalent to a point.

An example of homotopy equivalence that appears in Chapter 4 is in the context of suspensions. The **plain suspension** of X, denoted SX, is defined as $SX = (X \times [0,1])/(X \times \{0\})/(X \times \{1\})$. In essence, we take a cylinder over X and collapse each end to a point. The **reduced suspension** of X is obtained by further collapsing the line $\{x\} \times [0,1]$ for some $x \in X$, that is, $\Sigma(X) = SX/\{x\} \times [0,1]$.



(a) S^1 **(b)** $S^1 \times [0,1]$ **(c)** $S(S^1)$

Figure 2.3: Step-by-step sequence of taking the plain suspension of S^1.

As illustrated in Figure 2.9, it is easy to see that $S(S^1)$ is homotopy equivalent to S^2. More generally, it is well known that $S(S^n) \simeq S^{n+1}$. Additionally, for a CW complex X, we have that $\Sigma(X) \simeq S(X)$ (see [14, 0.10]).

Chapter 3

Maximal Simplices

In this chapter, we aim to determine the maximal simplices of each of the following complexes in terms of facets in $\mathrm{VR}(\mathbb{Z}^2; k)$:

1. $\mathrm{VR}(T_{n,n}; k)$ for any $k \geq 3$ and $n > 3k$,

2. $\mathrm{VR}(T_{3k,3k}; k)$ for any $k \geq 2$, and

3. $\mathrm{VR}(T_{3k-1,3k-1}; k)$ for any $k \geq 3$.

To begin, we prove some results pertaining to the construction of $T_{n,n}$ present in Definition 2.1.

Lemma 3.1. *Fix $n > 2k$. Suppose that $d(([a],[b]),([c],[d])) \leq k$. Then for any $(a',b') \in ([a],[b])$, there exists a unique $(c',d') \in ([c],[d])$ such that*

$$d((a',b'),(c',d')) = d(([a],[b]),([c],[d])) \leq k.$$

Proof. Fix $(a',b') \in ([a],[b])$. It is clear that there exists $(c',d') \in ([c],[d])$ such that $d((a',b'),(c',d')) = d(([a],[b]),([c],[d]))$, so we need only show uniqueness. By construction, any other $(c'',d'') \in ([c],[d])$ must be such that $d((c',d'),(c'',d'')) \geq n > 2k$. By the reverse triangle inequality,

$$d((a',b'),(c'',d'')) \geq |d((a',b'),(c',d')) - d((c',d'),(c'',d''))| > |k - 2k| = k.$$

So, (c',d') is unique. $\square$

Lemma 3.2. *Let σ be a facet in the complex* $\mathrm{VR}(\mathbb{Z}^2;k)$. *For $n > 2k+1$, $\pi_n(\sigma)$ is a facet in* $\mathrm{VR}(\pi_n(\mathbb{Z}^2);k)$.

Proof. We let x be an element in $\mathbb{Z}^2$ and $[x] = \pi_n(x)$. Clearly, $\pi_n(\sigma)$ is a simplex in $\mathrm{VR}(\pi_n(\mathbb{Z}^2);k)$. Suppose that $\pi_n(\sigma)$ is not maximal. Then pick $[y] \notin \pi_n(\sigma)$ such that $d([y],[x]) \leq k$ for all $x \in \sigma$. Then by Lemma 3.1, for each $x \in \sigma$, there is a unique $y_x \in [y]$ such that $d(x, y_x) \leq k$. If all of the y_x points are the same, then $\sigma \cup \{y_x\}$ is a simplex in $\mathrm{VR}(\mathbb{Z}^2;k)$, which contradicts with the maximality of σ. Otherwise suppose that there are more than one such y_x. Let A be the set containing all such y_x associated to some x. Since each $x \in \sigma$ must correspond to a unique y_x, we partition A into sets based on the associated y_x. It is easy to see that there can be no $x' \in \sigma$ such that $d(x', x) > 1$ for all $x \in \sigma$. Hence, for some two points, say $x', x'' \in \sigma$, we must have $d(x', x'') = 1$ while $y_{x'} \neq y_{x''}$. Then $d(y_{x'}, y_{x''}) \leq d(y_{x'}, x') + d(x', x'') + d(x'', y_{x''}) \leq 2k+1$, which contradicts the fact that $d(y_{x'}, y_{x''}) \geq n > 2k+1$. For a proof using the maximal simplices in $\mathrm{VR}(\mathbb{Z}^2;k)$ see [3]. $\square$

Lemma 3.3. *Let x and y be elements in $\mathbb{Z}^2$ with $\pi_n(B_{\mathbb{Z}^2}[x,k] \cap B_{\mathbb{Z}^2}[y,k]) = \pi_n(B_{\mathbb{Z}^2}[x,k]) \cap \pi_n(B_{\mathbb{Z}^2}[y,k])$. Then any facet τ in $\mathrm{VR}(\pi_n(\mathbb{Z}^2);k)$ containing $[x]$ and $[y]$ is in the form of $\pi_n(\sigma)$ for a facet $\sigma \in \mathrm{VR}(\mathbb{Z}^2;k)$ if one of the following holds:*

(i) $2 \leq d(x,y) \leq k$ *and* $n \geq 3k-1$, *or*

(ii) $1 \leq d(x,y) \leq k$ *and* $n \geq 3k$.

Proof. Suppose (i) holds and denote $B_{\mathbb{Z}^2}[x,k]$ by $B[x,k]$ for any $x \in \mathbb{Z}^2$ and $k \geq 0$. Fix a facet τ in $\mathrm{VR}(\pi_n(\mathbb{Z}^2);k)$ containing $[x]$ and $[y]$. We first show that $B_{\mathbb{Z}^2/G_n}[[x],k] = \pi_n(B[x,k])$ for each $x \in \mathbb{Z}^2$ when $k > 1$. Let $[z] \in B_{\mathbb{Z}^2/G_n}[[x],k]$. Then $d([x],[z]) \leq k$, so by Lemma 3.1, there exists some $z_x \in [z]$ such that $d(x, z_x) = d([x],[z]) \leq k$. Since $z_x \in B[x,k]$, then $[z] \in \pi_n(B[x,k])$, so $B_{\mathbb{Z}^2/G_n}[[x],k] \subseteq \pi_n(B[x,k])$. The reverse containment is immediate. Since $\tau \in \mathrm{VR}(\pi_n(\mathbb{Z}^2);k)$, then the above equivalence implies that $\tau \subseteq \pi_n(B[x,k]) \cap \pi_n(B[y,k])$.

Next, applying Lemma 3.1, for each $[z] \in \tau$, pick z_x to be the unique element in $[z]$ such that $d(x, z_x) \leq k$. Note that, since $d(x, y) \leq k$, $y_x = y$. Let $\sigma = \{x\} \cup \{z_x : [z] \in \tau \text{ and } z \neq x\}$. By construction, it is clear that $\tau = \pi_n(\sigma)$. We claim that σ is a facet in $\mathrm{VR}(\mathbb{Z}^2; k)$.

There are two important observations about the integer grid $\mathbb{Z}^2$ with the l^1-metric. By the triangle inequality, for any $z, z' \in \mathbb{Z}^2$, if $d(z, z') \leq 2k - 2$, then $d(z, z'') \geq k + 1$ for any $z'' \in \pi_n(z')$ with $z'' \neq z'$ since $d(z', z'') \geq n \geq 3k - 1$. Also for any $z, z' \in B[x, k] \cap B[y, k]$, we have $d(z, z') \leq 2k - 2$ by elementary geometry and the fact that $d(x, y) \geq 2$.

We now show that σ is a simplex in $\mathrm{VR}(\mathbb{Z}^2; k)$. First, we prove that $d(z_x, y) \leq k$ for any $z_x \in \sigma$. Suppose, for the sake of contradiction, there exists some $z_x \in \sigma$ such that $d(z_x, y) > k$. We know that $z_x \neq x$ as $d(x, y) \leq k$. Since $z_x \in \sigma$, then $[z] \in \tau$, so there must exists some $z_y \in [z]$ such that $d(z_y, y) = d([z], [y])$. By the triangle inequality, $n = d(z_x, z_y) \leq d(z_y, x) + d(x, z_x)$, which implies $d(z_y, x) \geq n - k \geq 3k - 1 - k = 2k - 1$. So, $z_y \notin B[x, k]$, and by assumption, $z_x \notin B[y, k]$. Moreover, it is clear that $z' \notin B[x, k] \cap B[y, k]$ for all $z' \in [z]$ as each z' has $d(z', z_x) \geq n$ and $d(z', z_y) \geq n$. Thus, $[z] \notin \pi_n(B[x, k] \cap B[y, k]) = \pi_n(B[x, k]) \cap \pi_n(B[y, k])$, contradicting that $\tau \subseteq \pi_n(B[x, k]) \cap \pi_n(B[y, k])$. Hence, $\sigma \subseteq B[x, k] \cap B[y, k]$. Next we show that $d(z_x, z'_x) \leq k$ for any $[z], [z'] \in \tau$. By the observations, $d(z_x, z'_x) \leq 2k - 2$. For any other element $z'' \in [z']$, we have $d(z_x, z'') \geq k + 1$. Notice that $d([z], [z']) \leq k$. Then, by Lemma 3.1, $d(z_x, z'_x) = d([z], [z'])$.

Lastly, we show that σ is maximal. Suppose, for the sake of contradiction, that there exists a point $z_0 \in \mathbb{Z}^2$ not in σ such that $d(z_0, z) \leq k$ for all $z \in \sigma$. Then $d([z_0], [z]) \leq k$, and since $n \geq 3k - 1$, $[z_0] \neq [z]$ for any $[z] \in \tau$. This contradicts the maximality of τ. $\square$

Since $T_{n,n}$ can be expressed as Cartesian product $C_n \square C_n$ equipped with the shortest path metric, we next classify the maximal simplices of $\mathrm{VR}(C_n; k)$, equivalently $\mathrm{Cl}(C_n^k)$. Geometrically, this is equivalent to classifying the maximal simplices taken from a cross-section of the tubular portion of $T_{n,n}$. In doing this, we determine two distinct types of maximal simplices:

1. $\sigma_i' = \{v_i, v_{i+k}, v_{i+2k}\}$, which are maximal in $\mathrm{VR}(T_{3k,3k}; k)$, and

2. $\sigma_i'' = \{v_i, v_{i+k}, v_{i+2k-1}, v_{i+2k}\}$, which are maximal in $\mathrm{VR}(T_{3k-1,3k-1}; k)$.

We first clarify some notation. Let K be a simplicial complex, and let $M(K)$ denote the collection of maximal simplices in K. Consider the cycle graph C_n with vertex set $\{v_0, v_1, \ldots, v_{n-1}\}$ where $v_i, v_j \in E(C_n)$ for $|i-j| \equiv 1 \mod n$. Next, for each $i \in \{0, 1, \ldots, n-1\}$, let $\sigma_i = \{v_i, v_{i+1}, \ldots, v_{i+k}\}$ where each indices are reduced modulo n as necessary. It is easy to verify that $\sigma_i \in M(\mathrm{VR}(C_n; k))$ for all i. Below, we classify the maximal simplices of $\mathrm{VR}(C_n; k)$ when $n > 3k$, $n = 3k$, and $n = 3k - 1$.

Lemma 3.4. *We have*

$$M(\mathrm{VR}(C_n; k)) = \begin{cases} \{\sigma_i : i = 0, \ldots, n-1\} & \text{for } n > 3k, \\ \{\sigma_i : i = 0, \ldots, n-1\} \cup \{\sigma_i' : i = 0, \ldots, n-1\} & \text{for } n = 3k, k \geq 2, \\ \{\sigma_i : i = 0, \ldots, n-1\} \cup \{\sigma_i'' : i = 0, \ldots, n-1\} & \text{for } n = 3k-1, k \geq 3, \end{cases}$$

where $\sigma_i' = \{v_i, v_{i+k}, v_{i+2k}\}$ and $\sigma_i'' = \{v_i, v_{i+k}, v_{i+2k-1}, v_{i+2k}\}$, with all indices taken modulo n.

Proof. When $n > 3k$, it is clear that the simplices of type σ_i are maximal. So, we need only to verify that there are no other facets.

Let $\sigma \in M(\mathrm{VR}(C_n; k))$. We define the arc of a vertex v_i in C_n as the set of $k+1$ consecutive vertices starting at v_i, i.e., $\{v_i, v_{i+1}, \ldots, v_{i+k}\}$. Fix a vertex $v_j \in \sigma$, and note that all vertices in σ must lie in $B_{C_n}[v_j, k]$. In this proof, for convenience, we use $B[v, k]$ to denote $B_{C_n}[v, k]$ for any vertex $v \in C_n$ and $k \geq 0$. Let $v_t \in \sigma$ be such that $d(v_t, v_{j-k}) = \min\{d(v_i, v_{j-k}) : v_i \in \sigma\}$. We claim that every vertex in σ must be contained in the arc of v_t. To verify this, we have two cases to check: $v_t = v_j$ and $v_t \neq v_j$.

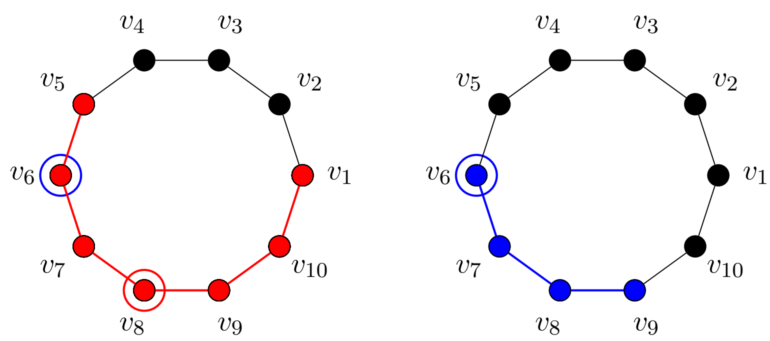


Figure 3.1: Visualization of $n = 10$ case where v_6 is the vertex in σ closest to $v_{j-k} = v_{8-3} = v_5$. The left diagram highlights $B[v_8, 3]$ in red, while the right diagram displays the arc of v_6 in blue.

Suppose $v_t = v_j$. For the sake of contradiction, assume that σ contains a vertex v_s not in the arc of v_t. Then $v_s \in B[v_j, k]$ and $v_s \notin \{v_{j-k}, v_{j-k+1}, \ldots, v_{j-1}\}$ as v_t is the closest point in σ to v_{j-k}. Hence, $v_s \notin B[v_j, k]$ as $v_s \notin \{v_{j-k}, v_{j-k+1}, \ldots, v_{j-1}\} \cup \{v_j, v_{j+1}, \ldots, v_{j+k}\}$, contradicting that $v_s \in B[v_j, k]$. Hence, σ must be of the form σ_i.

Suppose $v_t \neq v_j$. It is easy to see that $v_t \in \{v_{j-k}, \ldots, v_{j-1}\}$ as otherwise $d(v_t, v_{j-k}) > d(v_j, v_{j-k}) = k$. Again assume that σ contains a vertex v_s not in the arc of v_t. Since v_t is the vertex in σ closest to v_{j-k}, then $v_s \in B[v_j, k] \backslash \{v_{j-k}, v_{j-k+1}, \ldots, v_{t+k}\}$. We claim that $d(v_t, v_s) > k$. Note that $d(v_s, v_t)$ is the minimum of the distances from v_s to v_t measured in the clockwise and counterclockwise directions. The counterclockwise distance passing through v_j is clearly greater than k since v_s is not in the arc of v_t. Moreover, the clockwise distance is greater than k as the distance between the endpoints of $B[v_j, k]$ is minimally $k + 1$. Thus, in either case $d(v_t, v_s) > k$, contradicting that $v_s \in \sigma$. Consequently, all $\sigma \in M(\mathrm{VR}(C_n; k))$ must be of σ_i type.

When $n = 3k$, there are two types of facets. Denote $\sigma_i' = \{v_i, v_{i+k}, v_{i+2k}\}$, with all indices taken modulo n. We claim $M(\mathrm{VR}(C_n; k)) = \{\sigma_i : i = 0, 1, \ldots, n-1\} \cup \{\sigma_i' : i = 0, 1, \ldots, n-1\}$.

Once again, it is evident that each σ_i is maximal in this case.

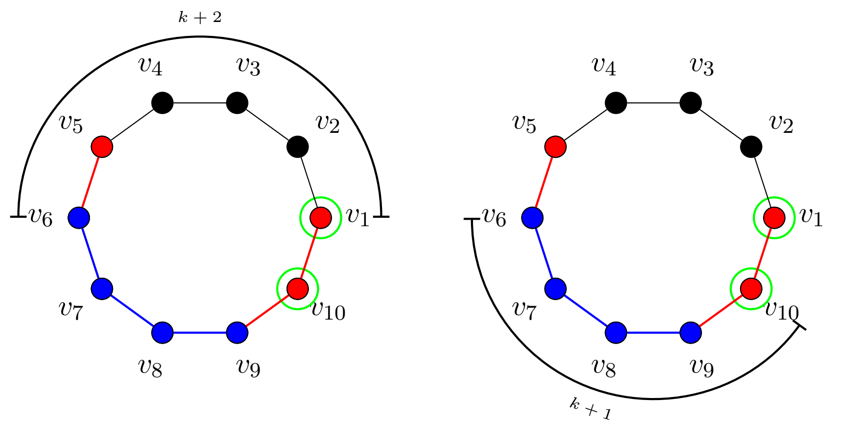


Figure 3.2: Continuation of the above case, illustrating the minimal distances in both the clockwise and counterclockwise directions from v_6 to vertices in $B[v_8, 3]$ that are not in the arc of v_6. These vertices are highlighted with green outlines for clarity.

We next show that each σ_i' is maximal. Let $\sigma_i' = \{v_i, v_{i+k}, v_{i+2k}\}$. Suppose that σ_i' is not maximal. Then it must be contained in a larger simplex, say τ. Let $v_j \in \tau \backslash \sigma_i'$. Then v_j is between some pair of vertices in σ_i'. Without loss of generality, suppose v_j is contained in the arc between v_i and v_{i+k}. It is clear that $d(v_j, v_{i+2k}) \geq k+1$, contradicting that $\tau \in \mathrm{VR}(C_n; k)$. Hence, the simplices of the form σ_i' are maximal.

Therefore, it suffices to show that any facet not of the form σ_i must be of the σ_i' type. Let $\sigma \in M(\mathrm{VR}(C_n; k))$, and fix a vertex $v_j \in \sigma$. Next, choose $v_t \in \sigma$ such that $d(v_t, v_{j-k}) = \min\{d(v_i, v_{j-k}) : v_i \in \sigma\}$. We claim that if $d(v_t, v_j) < k$, then σ is of the form σ_i. Otherwise, if $d(v_t, v_j) = k$, σ is either of type σ_i' or σ_i.

For the first assertion, if $0 < d(v_t, v_j) < k$, the argument is identical to the $n > 3k$ case. If instead $d(v_t, v_j) = 0$, then $\sigma \subseteq \{v_j, v_{j+1}, \ldots, v_{j+k}\}$, and since σ is maximal, σ must be of the form σ_i.

For the second assertion, suppose $d(v_t, v_j) = k$. We have the following two cases to check: $v_t = v_{j-k}$ or $v_t = v_{j+k}$.

If $v_t = v_{j+k}$, then $k = \min\{d(v_i, v_{j-k}) : v_i \in \sigma\}$. This forces σ to be of the form σ_i as $\sigma \subseteq \{v_j, v_{j+1}, \ldots, v_{j+k}\}$.

Suppose σ is not of the form σ_i, and let $v_s \in \sigma \backslash \{v_t, v_j\}$. If $v_t = v_{j-k}$, then we claim $v_s \in \{v_{j+1}, \ldots, v_{j+k}\}$. Suppose instead $v_s \in \{v_{j-k+1}, \ldots, v_{j-1}\}$. If $k = 2$, then $v_s = v_{j-1}$, which yields $\sigma = \{v_{j-2}, v_{j-1}, v_j\}$, a contradiction. If $k \geq 3$, then since σ is not of the form σ_i, simplex σ must contain a $v_i \in \{v_{j+1}, \ldots, v_{j+k}\}$. The only $v_i \in \{v_{j+1}, \ldots, v_{j+k}\}$ that is within k of both v_j and v_{j-k} is v_{j+k}. However, $d(v_s, v_{j+k}) > k$, a contradiction. Hence, $v_s \in \{v_{j+1}, \ldots, v_{j+k}\}$. Moreover, it is clear that $v_s = v_{j+k}$ as otherwise $d(v_{j-k}, v_s) > k$. Hence, σ is of type σ_i'.

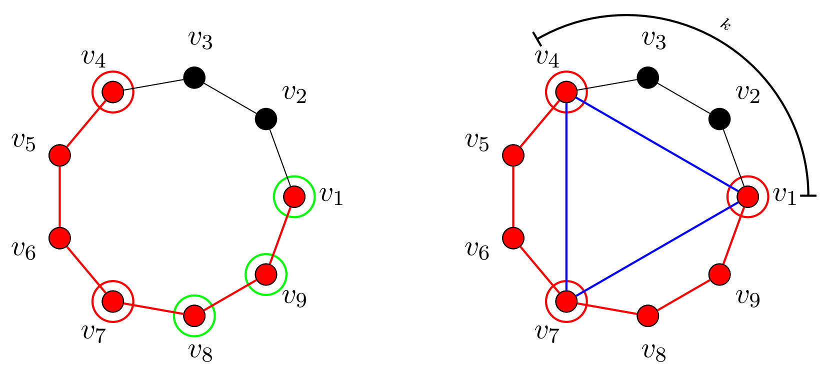


Figure 3.3: Visualization of $n = 9$ case where v_7 corresponds to v_j and v_4 corresponds to v_{j-k}. The left diagram highlights $B[v_7, 3]$ in red, along with all potential $v_s \in \{v_{j+1}, \ldots, v_{j+k}\}$, which are highlighted in green. The right diagram illustrates that v_1, which is v_{j+k} in this case, is the only vertex within k of v_4.

When $n = 3k - 1$, there are two types of facets. Denote $\sigma_i'' = \{v_i, v_{i+k}, v_{i+2k-1}, v_{i+2k}\}$, with all indices taken modulo n. We claim $M(\mathrm{VR}(C_n; k)) = \{\sigma_i : i = 0, 1, \ldots, n-1\} \cup \{\sigma_i'' : i = 0, 1, \ldots, n-1\}$.

The reasoning for this case follows a similar approach to that outlined in the $n = 3k$ case. So, we only show that the simplices of type σ_i'' are maximal.

Suppose that some σ_i'' is not maximal. Then σ_i'' must be contained in a larger simplex, say τ. Let $v_j \in \tau \backslash \sigma_i''$. Note that v_j must be on an arc between one of the following pairs: $\{v_i, v_{i+k}\}, \{v_{i+k}, v_{i+2k-1}\}$, or $\{v_i, v_{i+2k}\}$. If v_j is on the arc between $\{v_{i+k}, v_{i+2k-1}\}$, then $d(v_j, v_i) \geq k+1$. Similarly, if v_j is between $\{v_i, v_{i+2k}\}$, then $d(v_j, v_{i+2k-1}) \geq k+1$. Finally, if v_j is a vertex on the arc between $\{v_i, v_{i+k}\}$, we argue that either $d(v_j, v_{i+2k}) > k$ or $d(v_j, v_{i+2k-1}) > k$.

Suppose that $d(v_j, v_{i+2k}) \leq k$. Since $d(v_i, v_{i+2k}) = k-1$, $d(v_j, v_{i+2k}) = k$. Consequently, $d(v_j, v_{i+2k-1}) = k+1$, which contradicts the assumption that $\tau \in \mathrm{VR}(C_n; k)$. Therefore, the simplices of the form σ_i'' are maximal in this case. $\square$

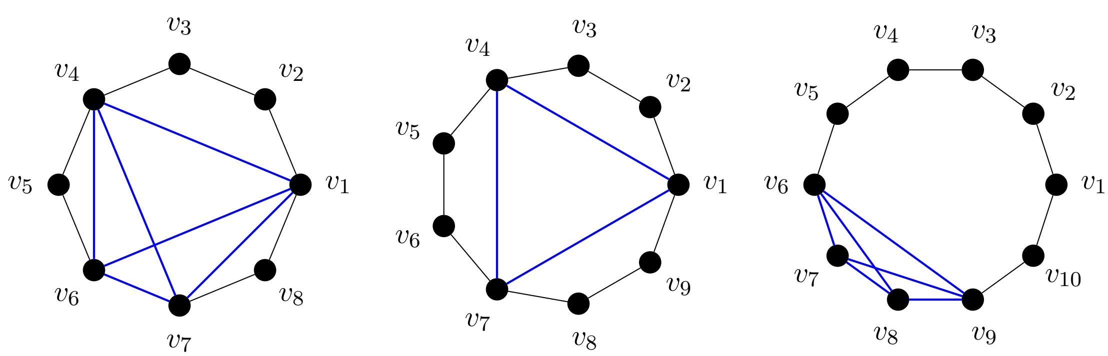


Figure 3.4: Example facets in $\mathrm{VR}(C_8; 3)$, $\mathrm{VR}(C_9; 3)$, and $\mathrm{VR}(C_{10}; 3)$

Lemma 3.5. *Fix different $x = (a, b)$ and $y = (a', b')$ in $\mathbb{Z}^2$ such that their distance is at most k. Then, $\pi_n(B_{\mathbb{Z}^2}[x, k] \cap B_{\mathbb{Z}^2}[y, k]) = \pi_n(B_{\mathbb{Z}^2}[x, k]) \cap \pi_n(B_{\mathbb{Z}^2}[y, k])$ if either of the following holds:*

(i) $n > 3k$.

(ii) $n \geq 3k-1$, $a \neq a'$, and $b \neq b'$.

Proof. To begin, let $B[x, k] = B_{\mathbb{Z}^2}[x, k]$. In both case (i) and (ii), the containment $\pi_n(B[x, k] \cap B[y, k]) \subseteq \pi_n(B[x, k]) \cap \pi_n(B[y, k])$ can be shown as follows. Let $[z] \in \pi_n(B[x, k] \cap B[y, k])$.

Then there exists some $z_i \in [z]$ with $z_i \in B[x,k] \cap B[x,k]$. Since $z_i \in B[x,k]$, then $[z] \in \pi_n(B[x,k])$. Similarly, $[z] \in \pi_n(B[y,k])$, so $\pi_n(B[x,k] \cap B[y,k]) \subseteq \pi_n(B[x,k]) \cap \pi_n(B[y,k])$. Below, we show the reverse containment for each case.

(i) Let $[z] \in \pi_n(B[x,k]) \cap \pi_n(B[y,k]) = B_{\mathbb{Z}^2/G_n}[[x],k] \cap B_{\mathbb{Z}^2/G_n}[[y],k]$. By Lemma 3.1, there exists unique elements $z_x, z_y \in [z]$ such that $d(z_x,x) \leq k$ and $d(z_y,y) \leq k$. We claim that $z_x = z_y$. If $z_x \neq z_y$, then, since z_x, z_y are distinct elements in $[z]$, we have $d(z_x,z_y) \geq n > 3k$. However, by the triangle inequality, $d(z_x,z_y) \leq d(z_x,x) + d(x,y) + d(z_y,y) \leq 3k$, so we must have $z_x = z_y$. Therefore, $z_x = z_y \in B[x,k] \cap B[y,k]$, which yields $[z] \in \pi_n(B[x,k] \cap B[y,k])$, as desired.

(ii) Let $[z] \in \pi_n(B[x,k]) \cap \pi_n(B[y,k]) = B_{\mathbb{Z}^2/G_n}[[x],k] \cap B_{\mathbb{Z}^2/G_n}[[y],k]$. As in (i), there exists unique elements $z_x, z_y \in [z]$ such that $d(z_x,x) \leq k$ and $d(z_y,y) \leq k$. We claim that $z_x = z_y$. Suppose $z_x \neq z_y$. Then $d(z_x,z_y) \geq n \geq 3k-1$, and $d(z_x,z_y) \leq d(z_x,x) + d(x,y) + d(z_y,y) \leq 3k$. So, either $d(z_x,z_y) = 3k-1$ or $d(z_x,z_y) = 3k$. We first prove the $d(z_x,z_y) = 3k-1$ case.

If $d(z_x,z_y) = 3k-1$, then $n \leq d(z_x,z_y) = 3k-1 \leq n$, so $n = 3k-1$. Since $d(z_x,z_y) = 3k-1$ and $n = 3k-1$, we have $z_x = z_y \pm (n,0)$ or $z_x = z_y \pm (0,n)$. Without loss of generality, suppose that $z_x = z_y + (n,0)$, i.e., z_x is a horizontal translation of z_y by n.

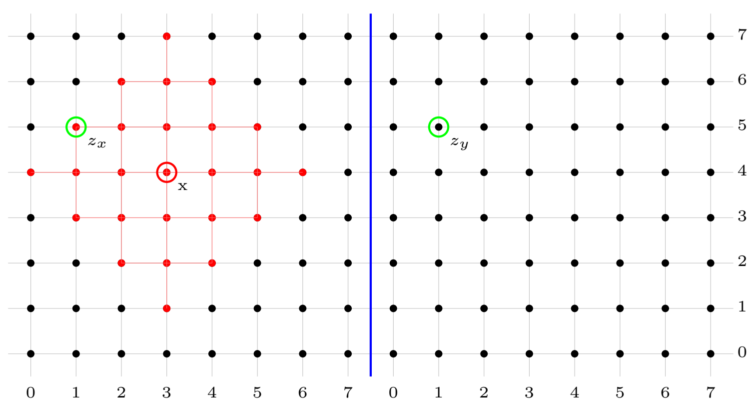


Figure 3.5: $B[x,k]$ in $k=3$ case with z_x and z_y.

Since $d(z_x, x) \leq k$ and $d(z_y, y) \leq k$, we have $z_x, z_y \in B[x, k] \cup B[y, k]$. We claim that the maximum distance between any two points on the same line in $B[x, k] \cup B[y, k]$ is $3k - 2$. To show this, we will maximize the distance between two points that lie on the same horizontal line; the vertical case follows similarly.

First, the diameter of $B[x, k] \cup B[y, k]$ is $2k$. To maximize the horizontal diameter of $B[x, k] \cup B[y, k]$ under the stipulation that $a \neq a'$, and $b \neq b'$, we let $y = (a + (k - 1), b + 1)$. Clearly, $B[y, k]$ extends the central horizontal line by $k - 2$ units. Hence, the maximum distance between two points that lie on the same horizontal line in $B[x, k] \cup B[y, k]$ is $2k + k - 2 = 3k - 2$.

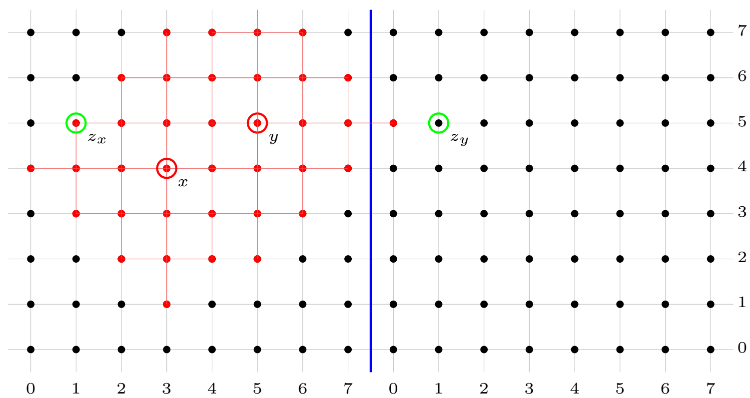


Figure 3.6: $B[x, k] \cup B[y, k]$ in $k = 3$ case with z_x and z_y.

However, since $z_x, z_y \in B[x, k] \cup B[y, k]$ and since z_x, z_y differ by a horizontal translation of length $3k - 1$, we have a contradiction. Hence, we must have $z_x = z_y$, which implies that $z_x = z_y \in B[x, k] \cap B[y, k]$. So, $[z] \in \pi_n(B[x, k] \cap B[y, k])$, as desired.

If $d(z_x, z_y) = 3k$, then $d(z_x, z_y) = 3k \geq n \geq 3k - 1$. Since $d(z_x, z_y)$ must be a multiple of n, then $n = 3k$. Repeating the argument above yields the desired result. $\square$

We let $M_{n,k}$ denote the collection of maximal simplices in $\mathrm{VR}(T_{n,n}; k)$ inherited from $\mathrm{VR}(\mathbb{Z}^2; k)$, i.e.

$$M_{n,k} = \{\pi_n(\sigma) : \sigma \text{ is a facet in } \mathrm{VR}(\mathbb{Z}^2; k)\}.$$

Additionally, we let $N_{3k,k}$ be the collection of triangles in the quotient space, i.e.

$$N_{3k,k} = \left\{ \begin{smallmatrix} \{([a],[b]),([a+k],[b]),([a+2k],[b])\} \\ \{([a],[b]),([a],[b+k]),([a],[b+2k])\} \end{smallmatrix} \ : \ 0 \le a, b \le 3k-1 \right\}.$$

Lemma 3.6. *For any $k \ge 2$, the collection of maximal simplices in* $\mathrm{VR}(T_{3k,3k}; k)$ *is* $M_{3k,k} \cup N_{3k,k}$.

Proof. Firstly, by lemma 3.2, $M_{3k,k} \subseteq M(\mathrm{VR}(T_{3k,3k}; k))$. Additionally, it is easy to verify that $N_{3k,k} \subseteq M(\mathrm{VR}(T_{3k,3k}; k))$, so $M_{3k,k} \cup N_{3k,k} \subseteq M(\mathrm{VR}(T_{3k,3k}; k))$.

Conversely, let $\tau \in M(\mathrm{VR}(T_{3k,3k}; k))$. We want to show that τ is in $M_{3k,k} \cup N_{3k,k}$. To do this, note that τ must satisfy exactly one of the following:

1. There exist $[x] = ([a], [b]), [y] = ([c], [d]) \in \tau$ such that $[a] \ne [c]$ and $[b] \ne [d]$;

2. All elements in τ have the same first or second coordinate.

To verify case (1), assume, without loss of generality, that $x = (a, b)$, $y = (c, d)$, and $d(x, y) = d([x], [y])$. Since $[a] \ne [c]$ and $[b] \ne [d]$, then $a \ne c$ and $b \ne d$. Moreover, this implies that $2 \le d(x, y) \le k$, so by Lemma 3.5, $\pi_n(B_{\mathbb{Z}^2}[x, k] \cap B_{\mathbb{Z}^2}[y, k]) = \pi_n(B_{\mathbb{Z}^2}[x, k]) \cap \pi_n(B_{\mathbb{Z}^2}[y, k])$. Then by Lemma 3.3, τ is of the form $\pi_n(\sigma)$ for some $\sigma \in M(\mathrm{VR}(\mathbb{Z}^2; k))$. Thus, $\tau \in M_{3k,k}$.

For case (2), without loss of generality, assume that all elements of τ have the same first coordinate, say $[a]$. Let L be the full subcomplex of $\mathrm{VR}(T_{3k,3k}; k)$ with vertices all having first entry $[a]$. Notice that L is isomorphic to the clique complex $\mathrm{Cl}(C_{3k}^k)$, where C_{3k} is a cyclic graph. By Lemma 3.4, $\mathrm{Cl}(C_{3k}^k)$ has two types of maximal simplices. It is easy to see that, of those two types, only those of the form $\{([a], [b]), ([a], [b+k]), ([a], [b+2k])\}$ for some $0 \le b \le 3k-1$ are maximal in $\mathrm{VR}(T_{3k,3k}; k)$. Hence, $\tau \in N_{3k,k}$, as desired. $\square$

We let $N_{3k-1,k}$ be the collection of tetrahedra in the quotient space, i.e.

$$N_{3k-1,k} = \left\{ \substack{\{([a],[b]),([a+k],[b]),([a+2k-1],[b]),([a+2k],[b])\} \\ \{([a],[b]),([a],[b+k]),([a],[b+2k-1]),([a],[b+2k])\}} \;:\; 0 \le a, b \le 3k-2 \right\}.$$

The following two lemmas are analogs of Lemma 3.6.

Lemma 3.7. *For any $k \ge 3$, the collection of facets in $\mathrm{VR}(T_{3k-1,3k-1}; k)$ is $M_{3k-1,k} \cup N_{3k-1,k}$.*

Lemma 3.8. *For $n > 3k$, the collection of facets in $\mathrm{VR}(T_{n,n}; k)$ is $M_{n,k}$.*

Chapter 4

Homotopy Types

In this chapter, we prove each of the following cases

1. $\mathrm{VR}(T_{n,n};k) \simeq \mathbb{T}^2$ for any $k \geq 3$ and $n > 3k$,
2. $\mathrm{VR}(T_{3k,3k};k) \simeq \bigvee_{6k^2-1} S^2$ for any $k \geq 2$, and
3. $\mathrm{VR}(T_{3k-1,3k-1};k) \simeq \bigvee_{6k-3} S^2 \vee \bigvee_{6k-2} S^3$ for any $k \geq 3$.

For case (1), an outline of our proof goes as follows. We first recall that, by Lemma 3.8, $M(\mathrm{VR}(T_{n,n};k)) = M_{n,k}$ when $n > 3k$ and $k \geq 3$. Next, using the homeomorphism $\mathbb{T}^2 \cong \mathbb{R}^2/G_n$ and the projection map $\pi_n : \mathbb{R}^2 \to \mathbb{R}^2/G_n$ as defined in Definition 2.1, we construct an open cover $\mathcal{U}$ of $\mathbb{T}^2$. Each open set in $\mathcal{U}$ is of the form $\pi_n(B_{\mathbb{R}^2}\left(x, \frac{k+1}{2}\right))$ for some $x \in \mathbb{Z}^2$. Using the identity $\pi_n\left(B_{\mathbb{R}^2}\left(x, \frac{k+1}{2}\right)\right) \cap \pi_n\left(B_{\mathbb{R}^2}\left(y, \frac{k+1}{2}\right)\right) = \pi_n\left(B_{\mathbb{R}^2}\left(x, \frac{k+1}{2}\right) \cap B_{\mathbb{R}^2}\left(y, \frac{k+1}{2}\right)\right)$, we then show that every nonempty finite intersection of sets in $\mathcal{U}$ is contractible. Consequently, Theorem 2.3 implies that the nerve $N(\mathcal{U})$ is homotopy equivalent to $\mathbb{T}^2$.

To relate this to the Vietoris–Rips complex, we use the following result from [3, Corollary 4.3]

Lemma 4.1. *Let $k \geq 0$ be an integer. Then $\sigma \subseteq \mathbb{Z}^2$ is a simplex in $\mathrm{VR}(\mathbb{Z}^2;k)$ if and only if $\cap_{v\in\sigma} B_{\mathbb{R}^2}(v, \frac{k+1}{2}) \neq \emptyset$.*

Using Lemma 4.1 in conjunction with the identity noted above, we conclude that $\mathrm{VR}(T_{n,n};k)$ is homotopy equivalent to $N(\mathcal{U})$, and hence, $\mathrm{VR}(T_{n,n};k) \simeq N(\mathcal{U}) \simeq \mathbb{T}^2$.

Theorem 4.2. *Suppose $k \geq 2$. For any $n > 2k$, let K be the complex with facets $M_{n,k}$. Then K is homotopy equivalent to the torus $\mathbb{T}^2$. Hence, when $k \geq 2$ and $n > 3k$,* $\mathrm{VR}(T_{n,n}; k)$ *is homotopy equivalent to the torus $\mathbb{T}^2$.*

Proof. Using a construction analogous to Definition 2.1, it is easy to see that $\mathbb{R}^2/G_n$ is homeomorphic to $\mathbb{T}^2$. Let $\pi_n : \mathbb{R}^2 \to \mathbb{R}^2/G_n$ be the canonical quotient map, and define the cover

$$\mathcal{U} = \left\{ \pi_n \left(B_{\mathbb{R}^2} \left(x, \frac{k+1}{2} \right) \right) \subseteq \mathbb{R}^2/G_n : x \in \mathbb{Z}^2 \right\}.$$

We claim that the intersection of any finite subcollection of $\mathcal{U}$ is either empty or contractible.

Firstly, notice that $B_{\mathbb{R}^2}(x, \frac{k+1}{2})$ has diameter $k+1$, and since $k \geq 2$, $n > 2k > k+1$. So, $\pi_n(B_{\mathbb{R}^2}(x, \frac{k+1}{2}))$ does not overlap itself when wrapped around the torus. From this, it's clear that $\pi_n(B_{\mathbb{R}^2}(x, \frac{k+1}{2}))$ is homeomorphic to $B_{\mathbb{R}^2}(x, \frac{k+1}{2})$.

Next, consider a finite subcollection $\mathcal{A} \subset \{B_{\mathbb{R}^2}(x, \frac{k+1}{2}) : x \in \mathbb{Z}^2\}$ with $\cap\mathcal{A} \neq \emptyset$. Since each element of $\mathcal{A}$ is open and convex, then $\cap\mathcal{A}$ is open and convex. Hence, $\cap\mathcal{A}$ is contractible. Moreover, since $n > 2k$ and $k \geq 2$, an argument analogous to Lemma 3.5 yields $\pi_n(B_{\mathbb{R}^2}(x, \frac{k+1}{2})) \cap \pi_n(B_{\mathbb{R}^2}(y, \frac{k+1}{2})) = \pi_n(B_{\mathbb{R}^2}(x, \frac{k+1}{2}) \cap B_{\mathbb{R}^2}(y, \frac{k+1}{2}))$. Thus, it is easy to see that $\cap\mathcal{A}$ is homeomorphic to $\pi_n(\cap\mathcal{A})$. To be explicit, the restriction of π_n to an $n \times n$ grid containing $\cap\mathcal{A}$ is a homeomorphism. So, if a finite subcollection of $\mathcal{U}$ is not empty, then it is contractible, as desired. Since $\cup\mathcal{U} = \mathbb{T}^2$, then by Theorem 2.3, the nerve complex $N(\mathcal{U})$ is homotopy equivalent to $\mathbb{T}^2$.

Now, we argue that $N(\mathcal{U})$ is homotopy equivalent to K. To see this, we recall that $M_{n,k} = \{\pi_n(\sigma) : \sigma \text{ is a facet in } \mathrm{VR}(\mathbb{Z}^2; k)\}$. Let τ be a maximal simplex in K. Then $\tau = \pi_n(\sigma)$ for some $\sigma \in M(\mathrm{VR}(\mathbb{Z}^2; k))$. By Lemma 4.1, $\sigma \in M(\mathrm{VR}(\mathbb{Z}^2; k))$ if and only if $\cap_{x\in\sigma} B_{\mathbb{R}^2}(x, \frac{k+1}{2})$ is non-empty. As noted above, for any finite subcollection $\mathcal{A} \subset \{B_{\mathbb{R}^2}(x, \frac{k+1}{2}) : x \in \mathbb{Z}^2\}$, we have that $\cap\mathcal{A}$ is homeomorphic to $\pi_n(\cap\mathcal{A})$. Thus, $\pi_n(\cap_{x\in\sigma} B_{\mathbb{R}^2}(x, \frac{k+1}{2}))$ is homeomorphic to $\cap_{x\in\sigma} B_{\mathbb{R}^2}(x, \frac{k+1}{2})$, so $\tau = \pi_n(\sigma)$ is a simplex in K if and only if $\pi_n(\cap_{x\in\sigma} B_{\mathbb{R}^2}(x, \frac{k+1}{2})) =$

$\cap_{x\in\sigma}\pi_n(B_{\mathbb{R}^2}(x, \frac{k+1}{2})) \neq \emptyset$. Hence, by the definition of the nerve complex (see Section 2.7), $N(\mathcal{U})$ is homotopy equivalent to K, so $K \simeq \mathbb{T}^2$.

Finally, when $n > 3k$, the collection of facets in the complex $\mathrm{VR}(T_{n,n}; k)$ is $M_{n,k}$ by Lemma 3.7. Therefore, in this case, $\mathrm{VR}(T_{n,n}; k)$ is homotopy equivalent to $\mathbb{T}^2$. $\square$

4.1 Homotopy types of $\mathrm{VR}(\mathbf{T_{3k,3k}}; \mathbf{k})$

In this section we obtain the homotopy types along the red diagonal in Table 1.1; specifically, we show that $\mathrm{VR}(T_{3k,3k}; k) \simeq \bigvee_{6k^2-1} S^2$ for $k \geq 2$. The path to this proof is as follows. We first recall from our classification of maximal simplices in Section 3 that $M(\mathrm{VR}(T_{3k,3k}; k)) = M_{3k,k} \cup N_{3k,k}$ (see Lemma 3.6). Theorem 4.2 establishes that $M_{3k,k}$ is homotopy equivalent to $\mathbb{T}^2$. Hence, to yield our result, it suffices to understand how the homotopy type of $\mathbb{T}^2$ changes as we attach the $6k^2$ simplices from $N_{3k,k}$.

In Lemma 4.4, we show that attaching two simplices from $N_{3k,k}$—one covering the central hole and another covering a longitudinal circle—yields a complex homotopy equivalent to S^2. Subsequently, in Theorem 4.5, we use Lemma 4.3 (see [12] for a proof) to successively affix the remaining $6k^2 - 2$ triangles from $N_{3k,k}$. Each attachment yields an additional wedge summand of S^2, and a simple induction completes the result.

Lemma 4.3. *Suppose a simplicial complex $K = K_1 \cup K_2$ satisfies that the inclusion maps $\imath_1 : K_1 \cap K_2 \to K_1$ and $\imath_2 : K_1 \cap K_2 \to K_2$ are both null-homotopic. Then*

$$K \simeq K_1 \vee K_2 \vee \Sigma(K_1 \cap K_2).$$

Lemma 4.4. *Assume that $k \geq 2$. Let*

$$\tau_1 = \{([0], [0]), ([k], [0]), ([2k], [0])\} \quad \textit{and} \quad \tau_2 = \{([0], [0]), ([0], [k]), ([0], [2k])\}.$$

Additionally, let K be the simplicial complex with $M(K) = M_{3k,k} \cup \{\tau_1, \tau_2\}$. Then K is homotopy equivalent to the sphere S^2.

Proof. Consider the space $\mathbb{R}^2/G_{3k}$. Let X be the cell complex obtained by gluing two disks, say c_1 and c_2, along the circles $(\mathbb{R} \times \{0\})/G^1_{3k}$ and and $(\{0\} \times \mathbb{R})/G^2_{3k}$, where $G^1_{3k} = 3k\mathbb{Z} \times \{0\}$ and $G^2_{3k} = \{0\} \times 3k\mathbb{Z}$. For clarity, the geometric object created here is a torus with its center hole and meridional hole covered by disks.

Next, let X_1 be $\mathbb{R}^2/G_{3k}$, and let X_2 be the image of c_1 and c_2 under their gluing maps, so $X = X_1 \cup X_2$. Since X_2 is the wedge of two contractible spaces, then X_2 is contractible. It is easy to see that X_2 is a closed subcomplex of X. Thus, by [14, Proposition 0.17], X is homotopy equivalent to X/X_2. This quotient collapses the two 1-cells of X leaving us with a 0-cell and 2-cell. This is precisely the cell complex of S^2. Hence, X is homotopy equivalent to S^2.

Let K_1 be the complex with $M(K_1) = M_{3k,k}$, and let K_2 be the complex with $M(K_2) = \{\tau_1, \tau_2\}$. Since $n = 3k > 2k$ and $k \geq 2$, then by Theorem 4.2, K_1 is homotopy equivalent to $\mathbb{T}^2$, so K_1 is homotopy equivalent to X_1. Additionally, since a disk is homeomorphic to a triangle with interior, then it is clear that X_2 is homeomorphic to K_2. Now, consider $K_1 \cap K_2$ and $X_1 \cap X_2$. It is clear that the intersection $X_1 \cap X_2$ results in the wedge of the boundary of both of our disks. Similarly, $K_1 \cap K_2$ is the wedge of the boundaries of our triangles. Thus, $K_1 \cap K_2$ and $X_1 \cap X_2$ are homotopy equivalent to $S^1 \vee S^1$; moreover, it is easy to see that $K_1 \cap K_2$ and $X_1 \cap X_2$ are homeomorphic.

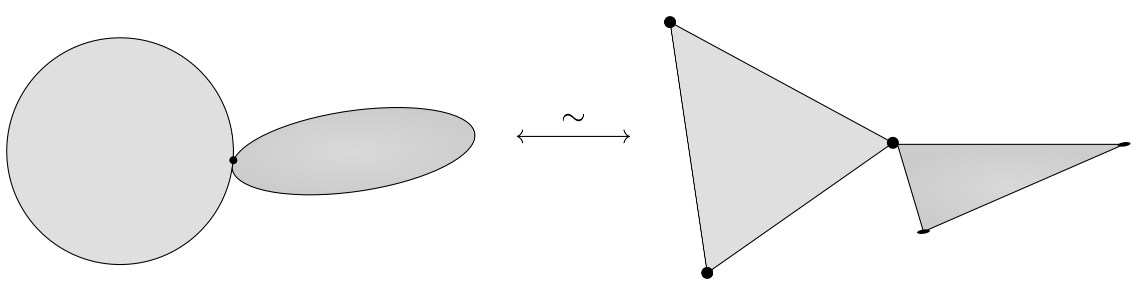


Figure 4.1: Homeomorphism between X_2 and K_2.

Since K_1 is homotopy equivalent to X_1, let $\varphi_1 : X_1 \to K_1$ be a homotopy equivalence. Similarly, fix a homeomorphism $\varphi_2 : X_2 \to K_2$. Note that, we may choose φ_1 and φ_2 such that $\varphi_1|_A = \varphi_2|_A = \varphi_0$ where φ_0 is a homeomorphism from A to L, which yields the following commutative diagram.

$$\begin{tikzcd}
X_1 \arrow[d, "\varphi_1"] & A \arrow[l, hook'] \arrow[r, hook] \arrow[d, "\varphi_0"] & X_2 \arrow[d, "\varphi_2"] \\
K_1 & L \arrow[l, hook'] \arrow[r, hook] & K_2
\end{tikzcd}$$

Since $A \hookrightarrow X_1$ and $L \hookrightarrow K_1$ are inclusions of CW-subcomplexes, then each of these maps is a closed cofibration. By [6, 7.5.7], we have the following commutative diagram where φ is a homotopy equivalence. Thus, since $X = X_1 \cup X_2$ is homotopy equivalent to S^2, then so is $K = K_1 \cup K_2$, as desired.

$$\begin{tikzcd}
A \arrow[rr, hook] \arrow[dr, "\varphi_1"] \arrow[dd, hook] & & X_2 \arrow[dr, "\varphi_2"] \arrow[dd, hook] & \\
& L \arrow[rr, hook] \arrow[dd, hook] & & K_2 \arrow[dd, hook] \\
X_1 \arrow[rr, hook] \arrow[dr, "\varphi_1"] & & X_1 \cup X_2 \arrow[dr, dashed, "\varphi"] & \\
& K_1 \arrow[rr, hook] & & K_1 \cup K_2
\end{tikzcd}$$

$\square$

Theorem 4.5. *For $k \geq 2$, we have the homotopy equivalence*

$$\mathrm{VR}(T_{3k,3k}; k) \simeq \bigvee_{6k^2-1} S^2.$$

Proof. By 3.6, we have that $M(\mathrm{VR}(T_{3k,3k}; k)) = M_{3k,k} \cup N_{3k,k}$. There are $6k^2$ distinct 2-simplices in $N_{3k,k}$. Moreover, it is easy to see that the intersection of any two simplices in $N_{3k,k}$ is at most a vertex.

Take the simplices $\sigma_{6k^2} = \{([0],[0]),([k],[0]),([2k],[0])\}$ and $\sigma_{6k^2-1} = \{([0],[0]),([0],[k]),([0],[2k])\}$ in $N_{3k,k}$, and let K be the complex such that $M(K) = M_{3k,k} \cup \{\sigma_{6k^2}, \sigma_{6k^2-1}\}$. By Lemma 4.4, K is homotopy equivalent to the sphere S^2. Denote the remaining elements of $N_{3k,k}$ as $\sigma_1, \sigma_2, \ldots, \sigma_{6k^2-2}$.

Let K_ℓ be the complex created by including the simplices $\sigma_1, \ldots, \sigma_\ell$ in K. We show that K_ℓ is homotopy equivalent to $\bigvee_{\ell+1} S^2$ by induction. Firstly, since K is homotopy equivalent to S^2, then the $\ell = 0$ case holds. Assume that $K_\ell \simeq \bigvee_{\ell+1} S^2$ for some $\ell < 6k^2 - 2$. Let $K_{\sigma_{\ell+1}}$ be the complex consisting of $\sigma_{\ell+1}$ and all its sub-faces. This yields that $K_{\ell+1} = K_\ell \cup K_{\sigma_{\ell+1}}$. Since $K_{\sigma_{\ell+1}}$ is a 2-simplex with boundary in K_ℓ, then $K_\ell \cap K_{\sigma_{\ell+1}}$ is homotopy equivalent to S^1. For clarification, by Theorem 4.2, $M_{3k,k}$ is homotopy equivalent to $\mathbb{T}^2$, and $K_{\sigma_{\ell+1}}$ corresponds to a disc covering either the central or meridional holes in $\mathbb{T}^2$, so $K_\ell \cap K_{\sigma_{\ell+1}} \simeq S^1$.

Next, we claim that $\imath_1 : K_\ell \cap K_{\sigma_{\ell+1}} \hookrightarrow K_\ell$ and $\imath_2 : K_\ell \cap K_{\sigma_{\ell+1}} \hookrightarrow K_{\sigma_{\ell+1}}$ are null-homotopic. $\imath_1$ is null-homotopic as $K_\ell \cap K_{\sigma_{\ell+1}}$ can be contracted along either σ_{6k^2} or σ_{6k^2-1}. Similarly, since $K_{\sigma_{\ell+1}}$ has interior, then $\imath_2$ is null-homotopic. Thus, by Lemma 4.3, $K_{\ell+1} \simeq K_\ell \vee K_{\sigma_{\ell+1}} \vee \Sigma(K_\ell \cap K_{\sigma_{\ell+1}}) \simeq K_\ell \vee K_{\sigma_{\ell+1}} \vee \Sigma(S^1) \simeq K_\ell \vee K_{\sigma_{\ell+1}} \vee S^2$. Finally, since $K_{\sigma_{\ell+1}}$ is contractible, then $K_{\ell+1} \simeq K_\ell \vee S^2 \simeq \bigvee_{\ell+2} S^2$, as desired.

$\square$

4.2 Homotopy types of $\mathrm{VR}(\mathbf{T_{3k-1,3k-1}};\mathbf{k})$

In this section we obtain the homotopy types along the teal diagonal in Table 1.1; specifically, we show that $\mathrm{VR}(T_{3k-1,3k-1}; k) \simeq \bigvee_{6k-3} S^2 \vee \bigvee_{6k-2} S^3$ for $k \geq 3$. The path to this proof mirrors the strategy discussed above in Section 4.1. We begin by recalling that, by Lemma 3.7, $M(\mathrm{VR}(T_{3k-1,3k-1}; k)) = M_{3k-1,k} \cup N_{3k-1,k}$ when $k \geq 3$. From Theorem 4.2, we know that $M_{3k-1,k}$ is homotopy equivalent to $\mathbb{T}^2$. So, the problem reduces to understanding how the homotopy type of $\mathbb{T}^2$ changes as we attach the tetrahedra from $N_{3k-1,k}$ (see Lemma 3.4).

In Lemma 4.7, we show that attaching two tetrahedra from $N_{3k-1,k}$—one covering the central hole and another covering a longitudinal circle—yields a complex homotopy equivalent to S^2. Next, as in Theorem 4.5, we use Lemma 4.3 to successively attach the $6k-2$ full subcomplexes of $\mathrm{VR}(T_{3k-1,3k-1};k)$ that correspond to the $3k-1$ longitudinal cycle graphs and $3k-1$ cycle graphs enclosing the central hole. Each attachment of some full subcomplex L contributes a wedge summand of $L \vee S^2$.

To determine the homotopy type of each such L, we use the following result of Adamaszek from [1, Corollary 6.7]

Lemma 4.6. *For any $n \geq 3$ and $0 \leq k < \frac{n}{2}$,*

$$\mathrm{VR}(C_n;k) \cong \begin{cases} \bigvee^{n-2k-1} S^{2\ell} & \text{if } k = \frac{\ell}{2\ell+1}n \\ S^{2\ell+1} & \text{if } \frac{\ell}{2\ell+1}n < k < \frac{\ell+1}{2\ell+3}n \end{cases}$$

where $\mathrm{VR}(C_n;k)$ denotes the clique complex of the k-th power fo the cycle graph C_n, as introduced in Section 2.2. In particular, when $n = 3k-1$, setting $\ell = 1$, this result yields that $L \simeq S^3$ when $k \geq 3$. Hence, each attachment of some full subcomplex L yields a wedge summand of $S^3 \vee S^2$, and a simple inductive argument yields the final result.

We omit the proof of the following lemma, as it is follows from an argument analogous to that of Lemma 4.4.

Lemma 4.7. *Assume that $k \geq 3$. Let $\tau_1 = \{([0],[0]),([k],[0]),([2k-1],[0]),([2k],[0])\}$ and $\tau_2 = \{([0],[0]),([0],[k]),([0],[2k-1]),([0],[2k])\}$ and let K be the simplicial complex with $M(K) = M_{3k-1,k} \cup \{\tau_1,\tau_2\}$. Then K is homotopy equivalent to the sphere S^2.*

Theorem 4.8. *For $k \geq 3$, we have the homotopy equivalence*

$$\mathrm{VR}(T_{3k-1,3k-1};k) \simeq \bigvee_{6k-3} S^2 \vee \bigvee_{6k-2} S^3.$$

Proof. For each $a \in \mathbb{Z}$, let L_a^1 be the full subcomplex of $\mathrm{VR}(T_{3k-1,3k-1}; k)$ with vertices having first coordinate $[a]$. Similarly, for each $b \in \mathbb{Z}$, let L_b^2 be the full subcomplex of $\mathrm{VR}(T_{3k-1,3k-1}; k)$ with vertices having second coordinate $[b]$. Notice that, for each a, the vertex set of L_a^1 corresponds to a cycle of $3k-1$ vertices on $\mathbb{T}^2$ enclosing the the meridional hole. Similarly, L_b^2 corresponds to a cycle of $3k-1$ vertices on $\mathbb{T}^2$ enclosing the central hole. From this, it is easy to see that the full subcomplexes L_a^1 and L_b^2 are isomorphic to $\mathrm{VR}(C_{3k-1}; k)$ (see Lemma 3.4).

By Lemma 4.6, $L_a^1 \simeq L_b^2 \simeq S^3$ when $k \geq 3$. Moreover, if $\mathcal{L} = \{L_a^1, L_b^2 : a, b \in \mathbb{Z}\}$, then it is easy to see that the intersection of any pair from $\mathcal{L}$ is at most a vertex. Let K be the simplicial complex with $M(K) = M_{3k-1,k} \cup \{\sigma_1, \sigma_2\}$ where $\sigma_1 = \{([0],[0]), ([0],[k]), ([0],[2k-1]), ([0],[2k])\}$ and $\sigma_2 = \{([0],[0]), ([k],[0]), ([2k-1],0), ([2k],[0])\}$. By Lemma 4.7, K is homotopy equivalent to S^2. Notice that we have

$$\mathrm{VR}(T_{3k-1,3k-1}; k) = K \cup \bigcup_{a \in \mathbb{Z}} L_a^1 \cup \bigcup_{b \in \mathbb{Z}} L_b^2.$$

Next, since any maximal simplex in $M_{3k-1,k}$ consists of at most k sequential vertices in the vertex set of L_a^1, then $K \cap L_a^1$ is a subcomplex with maximal simplices $\{([a],[b]), ([a],[b+2k-1]), ([a],[b+2k])\}$, $\{([a],[b]), ([a],[b+2k])\}$, and $\{([a],[b+k]), ([a],[b+2k]), ([a],[b+2k-1])\}$. Hence, when $a \neq 0$, $K \cap L_a^1$ is homotopy equivalent to S^1(See Figure 4.2).

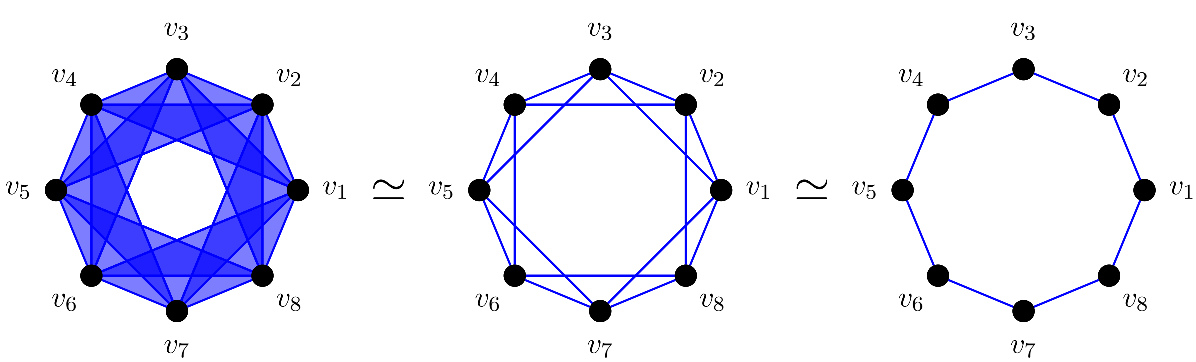


Figure 4.2: $K \cap L_a^1 \simeq S^1$ when $k = 2$.

When $a = 0$, it is easy to see that $K \cap L_0^1$ is contractible through σ_1. Thus, by Lemma 4.3, $K \cup L_0^1 \simeq K \vee L_0^1 \vee \Sigma(K \cap L_0^1) \simeq S^2 \vee S^3 \vee \Sigma(*) \simeq S^2 \vee S^3 \vee * \simeq S^2 \vee S^3$. Moreover, we claim that $K \cup L_0^1 \cup L_0^2 \simeq S^2 \vee S^3 \vee S^3$. Analogous to the above argument, $K \cup L_0^1 \cap L_0^2$ is contractible through σ_2. Hence, $K \cup L_0^1 \cup L_0^2 \simeq K \cup L_0^1 \vee L_0^2 \vee \Sigma(K \cup L_0^1 \cap L_0^2) \simeq S^2 \vee S^3 \vee S^3 \vee \Sigma(*) \simeq S^2 \vee S^3 \vee S^3$.

Let K_ℓ be the complex created by taking the union of $K \cup L_0^1 \cup L_0^2$ with ℓ unique elements from $\mathcal{L} \backslash \{L_0^1, L_0^2\}$. We claim that $K_\ell \simeq \bigvee_{1+\ell} S^2 \vee \bigvee_{2+\ell} S^3$. Firstly, since $K \cup L_0^1 \cup L_0^2 \simeq S^2 \vee S^3 \vee S^3$, then the $\ell = 0$ case holds. Next, suppose that our claim holds for K_ℓ, and let $L \in \mathcal{L} \backslash \{L_0^1, L_0^2\}$ be an element not in the union K_ℓ. Since the intersection of any pair in $\mathcal{L}$ is at most a vertex, it is easy to see that $K_\ell \cap L = K \cap L \simeq S^1$. Moreover, the inclusion $\imath_1 : K_\ell \cap L \hookrightarrow K_\ell$ is null-homotopic because $K_\ell \cap L$ can be contracted along either L_0^1 or L_0^2 depending on L's orientation, and $\imath_2 : K_\ell \cap L \hookrightarrow L$ is null-homotopic trivially. Hence, by Lemma 4.3, $K_\ell \cup L \simeq K_\ell \vee L \vee \Sigma(K_\ell \cap L) \simeq \bigvee_{1+\ell} S^2 \vee \bigvee_{2+\ell} S^3 \vee S^3 \vee \Sigma(S^1) \simeq \bigvee_{1+\ell} S^2 \vee \bigvee_{2+\ell} S^3 \vee S^3 \vee S^2 \simeq \bigvee_{1+(\ell+1)} S^2 \vee \bigvee_{2+(\ell+1)} S^3$.

Since $\mathcal{L} \backslash \{L_0^1, L_0^2\}$ contains $3k - 2 + 3k - 2 = 6k - 4$ elements, then by induction, we have that $\mathrm{VR}(T_{3k-1,3k-1}; k) \simeq \bigvee_{6k-3} S^2 \vee \bigvee_{6k-2} S^3$, as desired.

$\square$

A proof of the $k = 2$ case of the above theorem can be found in [3, Theorem 5.13].

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