1. If $r(t) = \langle 1, 5t^2, 4t\rangle$, find $\kappa(0)$ (i.e., the curvature at $t = 0$).

Proof. Recall that the curvature of a vector-valued function is

$$\kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}.$$

We have $\mathbf{r}'(t) = \langle 0, 10t, 4\rangle$ and $\mathbf{r}''(t) = \langle 0, 10, 0\rangle$. Thus,

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 10t & 4 \\ 0 & 10 & 0 \end{vmatrix} = \langle -40, 0, 0\rangle .$$

Consequently, $|\mathbf{r}'(t) \times \mathbf{r}''(t)| = 40$. Moreover, $|\mathbf{r}'(t)| = \sqrt{(10t)^2 + 4^2} = \sqrt{100t^2 + 16}$. Therefore,

$$\kappa(t) = \frac{40}{(100t^2 + 16)^{3/2}}.$$

At $t = 0$,

$$\kappa(0) = \frac{40}{16^{3/2}} = \frac{40}{64} = \frac{5}{8}.$$

□

Note A. When using the unit tangent vector formula

$$\kappa(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|},$$

do not substitute $t = 0$ into $\mathbf{T}(t)$ before differentiating. If we first compute $\mathbf{T}(0)$, then $\mathbf{T}(0)$ is just a single constant vector, and differentiating it would incorrectly give 0. We need the derivative of the function $\mathbf{T}(t)$ at $t = 0$, namely $\mathbf{T}'(0)$. So, in general, we must first compute $\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$, differentiate, and then substitute $t = 0$.

To see why early substitution gives the correct answer in this problem, first note that, by the quotient rule,

$$\mathbf{T}'(t) = \frac{|\mathbf{r}'(t)|\,\mathbf{r}''(t) - \mathbf{r}'(t)\dfrac{d}{dt}|\mathbf{r}'(t)|}{|\mathbf{r}'(t)|^2}$$

For this problem, $\mathbf{r}'(t) = \langle 0, 10t, 4\rangle$, so $|\mathbf{r}'(t)| = \sqrt{100t^2 + 16}$. Hence, $\frac{d}{dt}|\mathbf{r}'(t)| = \frac{100t}{\sqrt{100t^2+16}}$. Therefore, $\frac{d}{dt}|\mathbf{r}'(t)|\big|_{t=0} = 0$. When we evaluate the quotient-rule formula at $t = 0$, the second term in the numerator vanishes:

$$\mathbf{T}'(0) = \frac{|\mathbf{r}'(0)|\,\mathbf{r}''(0) - \mathbf{r}'(0)\cdot 0}{|\mathbf{r}'(0)|^2} = \frac{\mathbf{r}''(0)}{|\mathbf{r}'(0)|}.$$

Since $\mathbf{r}''(0) = \langle 0, 10, 0\rangle$ and $|\mathbf{r}'(0)| = 4$, we obtain $\mathbf{T}'(0) = \left\langle 0, \frac{5}{2}, 0\right\rangle$.

2. The level curves of

$$f(x,y) = \sqrt{x^2 + y^2 + 1} + x$$

are:

Proof. Let $f(x,y) = c$. Then $\sqrt{x^2+y^2+1} + x = c$, so $\sqrt{x^2+y^2+1} = c - x$. Squaring both sides gives $x^2 + y^2 + 1 = (c-x)^2$. Expanding,

$$x^2 + y^2 + 1 = c^2 - 2cx + x^2.$$

Canceling x^2 from both sides, $y^2 + 1 = c^2 - 2cx$. Hence, $y^2 = -2cx + c^2 - 1$. For $c > 0$, solving for x yields:

$$x = -\frac{1}{2c}y^2 + \frac{c^2-1}{2c}.$$

Recall that the standard form of a horizontal parabola is $x = a(y-k)^2 + h$. Therefore, the level curves are parabolas.

$\square$