

1. If $f(x, y, z) = x^2yz - xy^2 + 2xz^2$, then $\text{div}(\text{grad}(f))$ at $(1, 1, 1)$ is equal to:

Proof. Recall that $\text{grad}(f) = \langle f_x, f_y, f_z \rangle$. So,

$$\text{div}(\text{grad}(f)) = \nabla \cdot \langle f_x, f_y, f_z \rangle = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle f_x, f_y, f_z \rangle = f_{xx} + f_{yy} + f_{zz}.$$

Computing, we find that

$$f_{xx} = 2yz, \quad f_{yy} = -2x, \quad f_{zz} = 4x,$$

so at $(1, 1, 1)$

$$\text{div}(\text{grad}(f)) = 2(1)(1) - 2(1) + 4(1) = 4.$$

Thus, the correct option is C. □

2. Find the area of the part of the plane $3x + 2y + z = 6$ that is in the first octant.

Proof. We can parameterize the plane by $r(x, y) = \langle x, y, 6 - 3x - 2y \rangle$. To restrict to the first octant, we need $x, y \geq 0$ and (x, y) in projection of the plane in the xy -plane, i.e., $\{(x, y) : 3x + 2y \leq 6\}$.

Thus, $0 \leq x \leq 2$ and $0 \leq y \leq 3 - \frac{x}{2}$. So, via our parameterization, the surface integral becomes

$$\iint_S 1 dS = \int_0^2 \int_0^{3-\frac{x}{2}} |r_x \times r_y| dy dx.$$

Computing, we find that

$$r_x = \langle 1, 0, -3 \rangle \quad \text{and} \quad r_y = \langle 0, 1, -2 \rangle.$$

Thus,

$$r_x \times r_y = \det \begin{pmatrix} i & j & k \\ 1 & 0 & -3 \\ 0 & 1 & -2 \end{pmatrix} = 3i + 2j + k = \langle 3, 2, 1 \rangle,$$

so $|r_x \times r_y| = \sqrt{9 + 4 + 1} = \sqrt{14}$. Finally, we have that

$$\begin{aligned} \int_0^2 \int_0^{3-\frac{x}{2}} |r_x \times r_y| dy dx &= \int_0^2 \int_0^{3-\frac{x}{2}} \sqrt{14} dy dx = \int_0^2 \sqrt{14} y \Big|_0^{3-\frac{x}{2}} dx = \\ &= \sqrt{14} \int_0^2 \left(2 - \frac{x}{2}\right) dx = \sqrt{14} \left(2x - \frac{x^2}{4}\right) \Big|_0^2 = 3\sqrt{14}, \end{aligned}$$

so the correct option is D. □