1. For the level surface $3y^2z + xz^2 = 10$, use implicit differentiation to find $2\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}$ at $(1, -1, 2)$.

Proof.

$$3y^2z + xz^2 = 10, \qquad z = z(x, y).$$

Differentiate with respect to x (holding y constant):

$$\frac{\partial}{\partial x}(3y^2z) + \frac{\partial}{\partial x}(xz^2) = 0$$

$$3y^2\, z_x + \left(z^2 + x \cdot 2z\, z_x\right) = 0$$

$$(3y^2 + 2xz)z_x + z^2 = 0$$

$$z_x = -\frac{z^2}{3y^2 + 2xz}.$$

Differentiate with respect to y (holding x constant):

$$\frac{\partial}{\partial y}(3y^2z) + \frac{\partial}{\partial y}(xz^2) = 0$$

$$3(2y)z + 3y^2\, z_y + x \cdot 2z\, z_y = 0$$

$$6yz + (3y^2 + 2xz)z_y = 0$$

$$z_y = -\frac{6yz}{3y^2 + 2xz}.$$

At $(x, y, z) = (1, -1, 2)$:

$$3y^2 + 2xz = 3(1) + 2(1)(2) = 7,$$

$$z_x = -\frac{4}{7}, \qquad z_y = \frac{12}{7}.$$

Therefore,

$$2z_x + z_y = 2\left(-\frac{4}{7}\right) + \frac{12}{7} = \frac{4}{7},$$

which is option C. $\square$

2. Let $f(x,y) = xye^{xy}$, then the direction of steepest descent at $(2,3)$ is in the direction of the vector...

Proof. The direction of steepest descent at a point is given by $-\nabla f$.

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(xye^{xy}\right) = ye^{xy} + xy \cdot e^{xy} \cdot y = ye^{xy}(1+xy)$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(xye^{xy}\right) = xe^{xy} + xy \cdot e^{xy} \cdot x = xe^{xy}(1+xy)$$

Hence

$$\nabla f(x,y) = \langle ye^{xy}(1+xy),\ xe^{xy}(1+xy)\rangle .$$

At $(2,3)$, we have $xy = 6$, so $e^{xy} = e^6$ and $1 + xy = 7$. Therefore,

$$\nabla f(2,3) = \left\langle 3 \cdot e^6 \cdot 7,\ 2 \cdot e^6 \cdot 7 \right\rangle = \left\langle 21e^6,\ 14e^6 \right\rangle .$$

So the direction of steepest descent is

$$-\nabla f(2,3) = \left\langle -21e^6,\ -14e^6 \right\rangle .$$

Any nonzero scalar multiple gives the same direction; dividing by $7e^6$ gives $\langle -3, -2\rangle$, i.e. option D.

$\square$