1. Evaluate $\iint_R \frac{x^2y}{2+x^3} dA$ over the region $R = \{(x, y) : 1 \le x \le 2, 0 \le y \le 4\}$.

Proof.

$$\iint_R \frac{x^2y}{2+x^3} dA = \int_1^2 \int_0^4 \frac{x^2y}{2+x^3} dydx = \int_1^2 \left[\frac{x^2y^2}{2(2+x^3)}\right]_0^4 dx = \int_1^2 \frac{16x^2}{2(2+x^3)} dx.$$

Now, let $u = 2 + x^3$. Then $du = 3x^2dx$. Therefore,

$$\iint_R \frac{x^2y}{2+x^3} dA = \int_1^2 \frac{16x^2}{2(2+x^3)} dx = \int_3^{10} \frac{8}{3u} du = \frac{8}{3}[\ln(u)]_3^1 0 = \frac{8}{3}(\ln(10) - \ln(3)).$$

So, either A or D is a correct option. □

2. The extreme values of $f(x, y, z) = 3x + 2y + 6z$ with constraint $x^2 + y^2 + z^2 = 4$ are:

Proof. Let $g(x, y, z) = x^2 + y^2 + z^2$. Recall, via the Lagrange multipliers method, the extreme values of f must occur at solutions to the following system of equations:

$$\begin{cases} \nabla f = \lambda \nabla g \\ g(x, y, z) = 4. \end{cases}$$

Computing, we find that

$$\nabla f = \langle 3, 2, 6\rangle \quad \text{and} \quad \nabla g = \langle 2x, 2y, 2z\rangle,$$

so our system is

$$\begin{cases} 3 = \lambda 2x \\ 2 = \lambda 2y \\ 6 = \lambda 2z \\ x^2 + y^2 + z^2 = 4 \end{cases} \implies \begin{cases} \frac{3}{2} = \lambda x \\ 1 = \lambda y \\ 3 = \lambda z \\ x^2 + y^2 + z^2 = 4. \end{cases}$$

Since $\lambda \neq 0$, we can solve for each variable in terms of λ and then substitute into the constraint. From the above, $x = \frac{3}{2\lambda}$, $y = \frac{1}{\lambda}$, and $z = \frac{3}{\lambda}$. Substituting, we find that

$$\left(\frac{3}{2\lambda}\right)^2 + \left(\frac{1}{\lambda}\right)^2 + \left(\frac{3}{\lambda}\right)^2 = 4 \iff \frac{9}{4\lambda^2} + \frac{1}{\lambda^2} + \frac{9}{\lambda^2} = 4 \iff$$

$$\frac{1}{\lambda^2}\left(\frac{9}{4} + 1 + 9\right) = 4 \iff 4\lambda^2 = \frac{49}{4} \iff \lambda = \pm\sqrt{\frac{49}{16}} = \pm\frac{7}{4}.$$

So, we have two cases to check: $\lambda = \frac{7}{4}$ and $\lambda = -\frac{7}{4}$. If $\lambda = \frac{7}{4}$, then $x = \frac{12}{14} = \frac{6}{7}$, $y = \frac{4}{7}$, and $z = \frac{12}{7}$, so $\left(\frac{6}{7}, \frac{4}{7}, \frac{12}{7}\right)$ is a point of interest. Similarly, if $\lambda = -\frac{7}{4}$, then $x = -\frac{6}{7}$, $y = -\frac{4}{7}$, and $z = -\frac{12}{7}$, so $\left(-\frac{6}{7}, -\frac{4}{7}, -\frac{12}{7}\right)$ is a point of interest. Plugging these values in, we find that $f\left(\frac{6}{7}, \frac{4}{7}, \frac{12}{7}\right) = \frac{98}{7} = 14$

and $f\left(-\frac{6}{7}, -\frac{4}{7}, -\frac{12}{7}\right) = -14$, so the correct answer is A. $\square$